JOURNAL OF THE EXPERIMENTAL ANALYSIS OF BEHAVIOR 2000, **74,** 189–206 NUMBER 2 (SEPTEMBER)

SYNTHESIZING CONCURRENT INTERVAL PERFORMANCES

JAMES S. MACDONALL

FORDHAM UNIVERSITY

Concurrent schedules may be viewed as consisting of two pairs of stay and switch schedules, each pair associated with one of the alternatives. A stay schedule arranges reinforcers for staying and responding at one alternative, whereas the associated switch schedule arranges reinforcers for switching to the other alternative. In standard concurrent schedules, the stay schedule at each alternative is equivalent to the switch schedule at the other alternative. MacDonall (1999) exposed rats to one pair of stay and switch variable-ratio schedules and varied the response requirements across conditions. Combining results from symmetric pairs produced composite performances that were described by the generalized matching law. This outcome was noteworthy because the data were obtained from performances at two alternatives with contingencies that were functionally unrelated to each other. This result suggests that concurrent performances may consist of two unrelated performances that alternate as behavior moves between alternatives. The purpose of the present experiment was to extend those results to interval schedules. Rats were exposed to pairs of random-interval schedules, and across conditions their mean intervals were varied. When data from appropriately paired conditions were combined, the composite performances were consistent with the generalized matching law. In addition, the results supported two models of concurrent performances that were based on local variables at an alternative (behavior, and stay and switch reinforcers): a modified version of the contingency discrimination model (Davison & Jenkins, 1985) and the local model (MacDonall, 1999).

Key words: generalized matching law, contingency discrimination model, local model, melioration, random-interval schedule, lever press, rats

---

Choice is frequently investigated using concurrent schedules of reinforcement in which variable-interval (VI) schedules are associated with the alternative responses. The first parametric investigation of performance on concurrent schedules found that the allocation of behavior among the alternatives equaled the allocation of reinforcers (Herrnstein, 1961). A general formulation of this relation, which is the most common and robust description of concurrent performance, is known as the generalized matching relation (Baum, 1974),

$$\log\left(\frac{B_1}{B_2}\right) = a \log\left(\frac{R_1}{R_2}\right) + \log b. \qquad (1)$$

B_n is the number of responses or time at alternative n. R_n is the number of reinforcers obtained at alternative n. The fitted parameters are a, which is behavioral sensitivity to differences in numbers of reinforcers obtained at each alternative, and log b, which is bias towards one alternative that is not related to the allocation of reinforcers.

When contingencies on concurrent schedules are viewed from a more local perspective, they can be decomposed into two *pairs* of schedules, with each pair associated with an alternative (Houston & McNamara, 1981; MacDonall, 1998). Each pair consists of a stay schedule, which arranges reinforcers for staying and responding at the present alternative, and a switch schedule, which arranges reinforcers for switching to the other alternative. For example, in a concurrent random-interval (RI) 36-s RI 312-s schedule, the pair of schedules at the first alternative are RI 36 s and RI 312 s, both of which operate only while the subject is at the first alternative. The RI 36-s schedule arranges reinforcers for staying at the first alternative, and the RI 312-s schedule arranges reinforcers for switching to the second alternative. At the second alternative, the pair of schedules are RI 312 s and RI 36 s, both of which operate only while the subject is at the second alternative. The RI

Portions of these data were presented at the 1999 meeting of the Association for Behavior Analysis. Data collection was supported by a research grant from Fordham University. I am grateful to Jonathan Galente for building and maintaining the equipment. I thank AnnMarie Abernanty, Deidre Beneri, Lisa Femia, Margaret McNamara, Lisa Musorrafiti, Christina Ojar, and Natalie Peterson for assistance in data collection.

Address correspondence to the author, Department of Psychology, Fordham University, 441 E. Fordham Road, Bronx, New York 10458 (E-mail: jmacdonall@fordham.edu).

312-s schedule arranges reinforcers for staying at the second alternative, and the RI 36-s schedule arranges reinforcers for switching to the first alternative. As this example demonstrates, in the standard concurrent procedure the pairs of schedules are *symmetric*, that is, the stay schedule of each pair is equivalent to the switch schedule of the other pair. This analysis suggests that concurrent schedules may consist of two functionally unrelated but symmetric sets of contingencies that produce two functionally unrelated performances, one at each alternative. Of course, these performances are related to each other by changeover responding, which switches these otherwise unrelated contingencies and the performances they maintain by stopping one pair of schedules and restarting the other pair.

Evidence supporting the view that the performances and the contingencies at the alternatives are not functionally related to each other was provided by exposing rats to just one pair of stay and switch variable-ratio (VR) schedules and systematically varying the response requirements (MacDonall, 1999). Responses at one lever, the stay lever, earned stay and switch reinforcers by incrementing both ratio counters and obtained stay reinforcers when arranged by the stay schedule. The first response at the other lever, the switch lever, obtained switch reinforcers when arranged by the switch schedule. The first and subsequent responses at the switch lever did not increment either counter and did not change alternatives or stimuli. For example, when the stay schedule was VR 20 and the switch schedule was VR 80, responses at the main (i.e., stay) lever incremented both ratio counters. Responses at the main lever were immediately reinforced according to the VR 20. When main-lever responses arranged a reinforcer according to the VR 80, that reinforcer was held until a response at the switch lever delivered the held reinforcer. Responses at the switch lever never incremented either counter and did not change any stimuli.

There were two main findings. First, the performances at each alternative, run lengths and visit durations, were related to the numbers of reinforcers obtained from the stay and switch schedules. Run length was the number of responses at the main lever divided by the number of switches. Visit duration was the time at the main lever divided by the number of switches. Second, when conditions were arranged into symmetric pairs that if combined produce standard concurrent procedures, the resulting composite concurrent performances were described by the following variation of the generalized matching law:

$$\log\left(\frac{B_1/C_2}{B_2/C_1}\right) = a \log\left[\frac{(Rt_1/C_2) + (Rw_1/C_1)}{(Rt_2/C_1) + (Rw_2/C_2)}\right] + \log b. \quad (2)$$

This equation was modified for per-visit responding, time and reinforcer allocation, by first dividing the numerator and denominator of each side of Equation 1 by the frequency of switching out of that alternative. Dividing by the frequencies of switching out of the alternatives was effectively dividing by a constant because the frequencies of switching at the two alternatives were large (greater than 100) and must be within 1 of each other. Thus, the values of the ratios were essentially unchanged. Then, the reinforcers per visit were separated into stay reinforcers per visit and switch reinforcers per visit. Rt_n represents the number of reinforcers obtained for staying at alternative n, Rw_n represents the number of reinforcers obtained for switching to alternative n, C_n is the number of switches to alternative n, and the other symbols are as in the previous equation. Thus, in the numerator, Rt_1/C_2 is the number of reinforcers obtained for staying at Alternative 1 divided by the number of switches to Alternative 2, which ends a visit. Rw_1/C_1 is the number of reinforcers obtained for switching to Alternative 1 divided by the number of switches to Alternative 1. Similar substitutions apply to the denominator. Equation 2, the per-visit version of the generalized matching law, fitted the per-visit data. Recall that unlike data from standard concurrent procedures, the number of stay reinforcers per visit and the number of switch reinforcers per visit in the numerator and denominator were obtained during different sessions. These sessions did not necessarily follow each other, yet the generalized matching law described the composite data. It is in this sense, then, that the data analysis used by MacDonall (1999) is said to synthesize performance on concurrent VR

schedules that were programmed like concurrent VI schedules (MacDonall, 1988).

The preceding analysis supports the view that concurrent performances are composed of unrelated performances at the alternatives. The composite performances were based on performances maintained by VR schedules, which are rarely used in investigations of concurrent performance. In a related experiment, when the performances maintained by functionally unrelated pairs of VI schedules were combined, the composite concurrent performances were described by the generalized matching law, but with large biases (MacDonall, 1998). These biases probably resulted from the fact that the procedures were only approximations of concurrent contingencies. The switch schedules ran continuously, rather than only when the stay schedule operated, as is the case under standard concurrent VI schedules. Accordingly, one purpose of the present experiment was to use stay and switch interval schedules arranged according to concurrent interval contingencies and to fit the generalized matching law to the composite concurrent performances.

Although the generalized matching law describes the composite concurrent performances that were maintained by functionally unrelated pairs of stay and switch schedules, it is not clear that the generalized matching law can describe the orderly performances at the alternatives, for example, the run lengths and visit durations. Ideally, one model can describe both the orderly performances at the alternatives and the concurrent performances. Therefore, the present research focused on two additional models: the local model (MacDonall, 1999), which is based on contingencies at the alternative, and a model related to the contingency discrimination model of Davison and Jenkins (1985).

The Local Model

The local model, as applied to the performances at the alternatives, will be described first, followed by its extension to concurrent performance. According to the local model, the performances at each alternative, namely run lengths and visit durations, are power functions of the ratio of the likelihoods of reinforcement for staying at that alternative divided by the likelihood of reinforcement for switching to the other alternative (MacDonall, 1999). Stay reinforcers are earned and obtained by behavior at the same alternative. Switch reinforcers are earned by behavior at one alternative and obtained by behavior at the other alternative (Green, Rachlin, & Hanson, 1983). For the local model, the behavior that earns the reinforcer is just as important as the behavior that obtains the reinforcer. This differs from the generalized matching law, which focuses exclusively on obtained reinforcers regardless of the behavior that earned them. The likelihood of reinforcement is, when using interval schedules, the rate of *earning* stay or switch reinforcers. The likelihood of reinforcement for staying at Alternative 1 is the number of reinforcers obtained for staying and responding at Alternative 1 divided by the time at Alternative 1, which earned those stay reinforcers. The likelihood of reinforcement for switching to Alternative 2 is the number of reinforcers obtained for switching to Alternative 2 divided by the time at Alternative 1, which earned those switch reinforcers. At Alternative 1, this is expressed as

$$\log\left(\frac{B_1}{C_2}\right) = k_1 \log\left(\frac{Lt_1}{Lw_2}\right) + \log\ m_1. \quad (3)$$

Lt_1 is the likelihood of reinforcement for staying and responding at Alternative 1. Lw_2 is the likelihood of reinforcement for switching to Alternative 2. There are two reasons why likelihood of reinforcement is used instead of probability of reinforcement, which was used previously (MacDonall, 1999). First, probability is inappropriate given the time-based schedules used in the present experiment. Second, likelihood is a more general term that can be understood as either probability or rate, depending on whether ratio or interval schedules are used. The fitted parameters are k_1, which is behavioral sensitivity to differences in the likelihoods of stay and switch reinforcement, and log m_1, which is the tendency to stay at the present alternative (MacDonall, 1999). The other symbols are as in previous equations. This equation says that the logs of the run lengths and visit durations are proportional to the logs of the ratios of the likelihoods of earning stay and switch reinforcement.

A model of concurrent performance can be derived from the local model, Equation 3

(see the Appendix for the derivation). This model also fitted the composite concurrent VR performances obtained by MacDonall (1999). The model is

$$\log\left[\frac{\left(\frac{B_1}{C_2}\right)}{\left(\frac{B_2}{C_1}\right)}\right] = k' \log\left(\frac{Lt_1 \cdot Lw_1}{Lt_2 \cdot Lw_2}\right) + \log m'. \quad (4)$$

The fitted parameters are log m', which is the bias towards one alternative and k', which is the behavioral sensitivity to differences in likelihood of earning stay and switch reinforcers. The other symbols are as in previous equations. Equation 4 says that the log of the ratio of the run lengths or visit durations is proportional to the log of the ratio of the products of the likelihoods of reinforcement at each alternative.

The Local Contingency Discrimination Model

After reviewing the contingency discrimination model (Davison & Jenkins, 1985), and showing how that model cannot apply to composite concurrent performances, a related model will be developed starting with performances at the alternatives and then extending it to composite concurrent performances.

The contingency discrimination model posits that behavior is perfectly sensitive to *perceived* reinforcer contingencies. However, occasionally reinforcer contingencies are misperceived, which produces undermatching (Davison & Jenkins, 1985; Davison & Jones, 1995). Contingencies are misperceived when reinforcers from Alternative 1 are misperceived as Alternative 2 reinforcers and when Alternative 2 reinforcers are misperceived as Alternative 1 reinforcers. The contingency discrimination model may be expressed as

$$\frac{B_1}{B_2} = n\left(\frac{R_1 - pR_1 + pR_2}{R_2 - pR_2 + pR_1}\right). \quad (5)$$

One fitted parameter p, which is the difficulty of correctly perceiving reinforcer contingencies, varies between 0 and .5. At 0, the reinforcer contingencies are perfectly discriminated, and at .5, reinforcer contingencies are completely confused, that is, random. The parameter n is bias towards either alternative that is not related to reinforcer allocation. The other symbols are as in previous equations.

The contingency discrimination model, Equation 5, cannot be applied to composite concurrent performances maintained by functionally unrelated pairs of stay and switch schedules, such as those used previously (MacDonall, 1998, 1999) and in the following experiment. When performances from functionally unrelated pairs of schedules are combined, the contingency misperception must be among contingencies that could, in fact, be misperceived. It is reasonable to model misperceptions between stay and switch reinforcers in the same session. It is not reasonable, however, to model misperceptions between stay and switch reinforcers delivered in different sessions, possibly widely separated in time. Said differently, because the composite performances were produced by two pairs of schedules that do not alternate within the same session, a contingency discrimination analysis must not require reinforcers allocated by one pair of schedules in one session to be confused with reinforcers allocated by the other pair of schedules in different sessions. Yet the contingency discrimination model, Equation 5, requires misperception among reinforcers from different pairs of schedules. This problem is eliminated when stay reinforcers at each alternative are misperceived only with switch reinforcers arranged by the paired switch schedule. Similarly, switch reinforcers at each alternative are misperceived only with stay reinforcers arranged by the paired stay schedule. A model of performance at each alternative emerges from the notion that the stay and switch reinforcers are important but that reinforcer allocation within a pair of schedules can be misperceived. This produces

$$\frac{B_1}{C_2} = n_1 \frac{(Rt_1/C_2 - p_1 Rt_1/C_2 + p_1 Rw_2/C_2)}{(Rw_2/C_2 - p_1 Rw_2/C_2 + p_1 Rt_1/C_2)}, \quad (6)$$

where the symbols are as in previous equations. This equation says that run lengths and visit durations are proportional to the ratio of the perceived number of reinforcers for staying and responding at Alternative 1 divided by the perceived number of reinforcers for switching to Alternative 2. The number of

perceived stay reinforcers are the actual number of stay reinforcers less a proportion, p, of the stay reinforcers that are perceived as switch reinforcers, plus the same proportion, p, of the switch reinforcers that are perceived as stay reinforcers. Similarly, the number of perceived switch reinforcers are the actual number of switch reinforcers less a proportion, p, of the switch reinforcers that are perceived as stay reinforcers, plus the same proportion, p, of the stay reinforcers that are perceived as switch reinforcers.

A model of concurrent performance emerges by taking the ratio of Equation 6 as applied to each alternative, letting $n' = n_1/n_2$, assuming $p_1 = p_2$, and substituting p'. This produces

$$\frac{B_1/C_2}{B_2/C_1} = n'\left(\frac{(Rt_1/C_2 - p'Rt_1/C_2 + p'Rw_2/C_2)}{(Rw_2/C_2 - p'Rw_2/C_2 + p'Rt_1/C_2)} \times \frac{(Rw_1/C_1 - p'Rw_1/C_1 + p'Rt_2/C_1)}{(Rt_2/C_1 - p'Rt_2/C_1 + p'Rw_1/C_1)}\right), \quad (7)$$

where the symbols are as in previous equations. Essentially, this model says that the ratio of run lengths and visit durations is a function of the ratio of the per-visit perceived number of stay reinforcers multiplied by the per-visit perceived number of switch reinforcers at each alternative.

The purpose of the present experiment was to determine whether composite concurrent interval performances maintained by functionally unrelated pairs of stay and switch schedules would be described by the generalized matching relation (Equation 2), the local model (Equation 4), and the local contingency discrimination model (Equation 7). Because VI schedules with a fixed set of interreinforcement intervals produce only an approximation to a random distribution of reinforcers, RI schedules were used. The design was similar to that used previously (MacDonall, 1999). Rats were exposed to pairs of stay and switch RI schedules. The values of the RI schedules were varied across conditions. At the end of the experiment, performances from each condition were arranged with its symmetric condition producing composite concurrent performances, which in turn were analyzed according to the three models.

METHOD

Subjects

The subjects were 4 female albino Sprague-Dawley rats obtained from Hilltop Lab Animals Inc. and maintained at 85% of their free-feeding weights. They were approximately 100 days old when the experiment began and were housed individually in a temperature-controlled colony room on a 14:10 hr light/dark cycle with free access to water.

Apparatus

Four operant conditioning chambers were used. Each was 20 cm wide and 20 cm high; three were 20 cm long, and one was 30 cm long. The food cup was centered horizontally on one wall (20 cm wide), and the center of the opening (5 cm square) was 3.5 cm above the floor. The centers of two response levers, 5 cm wide, 1 cm thick, and protruding 1.5 cm into the chamber (Gerbrands Model G6312), were located 5.5 cm from either side of the vertical center line of the food cup and 6.5 cm above the floor. A minimum force of approximately 0.3 N operated each lever. A Gerbrands feeder, located behind the food cup, dispensed 45-mg Noyes Formula A/1 rodent pellets. Each chamber was located in a sound- and light-controlled enclosure. A houselight mounted on the ceiling of the enclosure operated continuously during sessions. A speaker mounted on the ceiling of the chamber presented white noise during sessions. Located in an adjacent room was an IBM-compatible computer with hardware (MED Associates) and software (MED-PC®) that controlled contingencies and recorded responses.

Procedure

Rats were exposed to a pair of RI schedules. The first press on the left lever, the stay lever, started stay and switch RI schedules. Presses on the stay lever were occasionally reinforced according to the stay schedule. Presses on the right lever, the switch lever, stopped both RI schedules and were occasionally reinforced according to the switch

Table 1

For each rat, the sequence of conditions, the probability the random generator produced an output for stay and switch reinforcement, number of sessions, sums over the last 5 days of the number of stay responses, times in stay before a switch response, times in switch before a stay response, numbers of stay and switch reinforcers, and the number of switches. In all conditions the probability generator was sampled every 0.5 s.

		Probability ($\times 10^{-4}$) a reinforcer was arranged for				Time (s) switched		Reinforcers for		
Rat	Order	Staying	Switching	Sessions	Stay responses	In	Out	Staying	Switching	Switches
441	1	16	140	17	5,356	14,107.3	22,843.9	47	233	2,913
	2	78	78	26	16,251	18,335.3	12,839.7	270	229	2,530
	3	140	16	20	7,788	15,867.2	3,281.2	444	57	634
	4	134	22	13	7,763	19,921.9	4,362.3	420	81	828
	5	94	62	10	8,372	15,215.4	6,599.9	305	197	1,653
	6	78	78	11	8,299	15,967.0	10,679.7	257	243	2,246
	7	62	94	12	7,711	16,265.3	17,231.3	180	272	2,362
	8	22	134	12	8,899	14,092.2	21,796.9	57	337	3,784
442	1	140	16	22	7,134	16,571.4	5,397.6	441	60	774
	2	125	31	15	5,815	17,966.5	6,847.5	392	110	1,327
	3	78	78	13	2,962	17,043.2	6,674.5	256	245	1,634
	4	31	125	11	6,586	16,102.2	19.765.0	81	351	3,353
	5	16	140	10	5,650	9,003.2	26,901.8	23	209	2,598
	6	62	94	16	5,806	17,737.5	17,775.4	184	282	3,258
	7	39	117	12	4,279	14,379.1	21,414.8	100	287	2,833
	8	78	78	16	4,341	17,811.4	12,276.8	246	254	2,981
	9	94	62	12	4,000	18,135.1	10,728.5	300	200	2,666
	10	117	39	12	3,494	17,807.2	7,910.8	351	150	2,117
443	1	39	117	32	21,614	17,982.0	12,579.0	130	358	5,250
	2	94	62	13	20,164	17,760.5	7,985.1	313	188	4,301
	3	117	39	22	20,657	16,505.6	3,683.9	365	135	2,157
	4	78	78	23	22,233	17,117.9	6,131.4	254	247	2,681
	5	31	125	10	24,902	16,963.1	10,161.0	103	397	5,085
	6	62	94	15	27,118	17,834.3	8,430.8	201	300	4,602
	7	125	31	19	23,531	16,800.0	3,399.7	395	106	2,241
	8	78	78	10	21,886	16,296.2	5,849.9	257	244	2,926
444	1	78	78	36	16,022	16,937.7	5,119.2	261	239	1,343
	2	62	94	17	14,890	16,538.8	6,683.9	220	280	1,754
	3	94	62	22	15,890	16,212.5	4,255.0	275	225	1,996
	4	125	31	26	15,366	15,438.0	2,253.3	389	112	1,303
	5	26	130	11	14,583	17,586.4	11,116.2	106	394	3,643
	6	78	78	20	14,073	17,485.8	5,141.2	259	241	2,333
	7	130	26	13	15,110	15,571.6	1,523.2	429	76	927
	8	31	125	11	14,674	16,190.7	8,195.6	104	398	2,440

schedule. The next press on the stay lever restarted both schedules and collected a reinforcer, if arranged by the stay schedule. This happened when the stay schedule arranged a reinforcer after the last press on the stay lever and before the switch lever was pressed. When an RI schedule arranged a reinforcer, that RI schedule stopped until a press of the appropriate lever delivered that reinforcer. The other RI schedule continued operating. Because a changeover delay would alter the stay and switch contingencies, a changeover delay was not used.

Both RI schedules sampled a probability generator every 0.5 s. After an output from the generator occurred, the next press on the appropriate lever was reinforced. Across conditions the probability of an output varied, with the sum of the stay and switch probabilities constant. This kept the overall scheduled likelihood of reinforcement, the sum of the stay and switch likelihoods, approximately constant at .0156 every 0.5 s, which produced an overall RI of 32 s. Schedules ranged from RI 35.7 s to RI 312.5 s.

Table 1 lists, for each rat, the arrangement

of schedules in each condition, the sequence of conditions, and the number of sessions that each condition was in effect. Rats were exposed to 8 or 10 conditions. A condition remained in effect for at least 10 sessions and until visual inspection showed that there were no apparent upward or downward trends in run lengths and visit durations for five consecutive sessions. Sessions were typically conducted 6 days per week and ended after the first changeover response following the 100th reinforcer.

Initial training. First, each rat was trained to approach the food cup at the sound of the food dispenser operating. Then, pressing the stay lever was shaped by the method of successive approximations. After approximately 40 reinforced presses on the stay lever, rats were exposed to Condition 1. Pressing the switch lever was not shaped. This response emerged when reinforcement for stay-lever responses declined with the introduction of the schedules in the first condition.

RESULTS

All analyses were based on the sums of the data from the last 5 days of each condition. Table 1 presents these sums for numbers of responses at the stay lever, durations of visits at the stay lever, times between visits at the stay lever, numbers of stay and switch reinforcers, and numbers of switches. The duration of visits at the stay lever, called ''switched in'' time in Table 1, was the cumulative duration from the first press of the stay lever, which started a visit, to the first press of the switch lever, which ended a visit. The duration between visits at the stay lever, called ''switched out'' time in Table 1, was the cumulative duration from the first press of the switch lever to the first press of the stay lever.

The first analyses assess how well the generalized matching law describes the composite concurrent performances. Conditions were grouped into symmetric pairs that, when combined, form standard concurrent procedures. For example, the pair of schedules RI x s for staying and RI y s for switching and the pair of schedules RI y s for staying and RI x s for switching are symmetric pairs of schedules, and when combined form concurrent RI x-s RI y-s schedules. The data from these paired conditions were combined according to Equation 2. For each pair of conditions, the data from the condition that the rat was first exposed to was considered to be data from Alternative 1 in Equation 2. Similarly, the data from the second of the paired conditions was considered to be data from Alternative 2. Then per-visit run lengths, visit durations, and stay and switch reinforcers were obtained. Run length was the total number of stay responses divided by the number of switches. Visit duration was the total time between the first stay response and the next switch response (switched in time) divided by the number of switches. Per-visit data, rather than the sums, were used because each condition produced different numbers of switches.

Figures 1 and 2 show, for each rat, that the log of the ratio of run lengths and the log of the ratio of visit durations increased approximately linearly with the log of the composite reinforcer ratio. The composite reinforcer ratio was the sum of the per-visit stay and switch reinforcers at one alternative divided by the sum of the per-visit stay and switch reinforcers at the other alternative. The sum of the per-visit stay and switch reinforcers at an alternative was calculated according to the numerator and denominator of Equation 2. The solid lines represent the best fit of Equation 2 to these data using the least squares method. Table 2 presents these fits, which, with the exception of run lengths for Rat 443, are good to excellent ($r^2 > .81$). There is no consistent bias. The slopes of run length and visit duration for Rat 441 reflect slight undermatching ($a < 1.0$), whereas the slopes for the other six functions show large undermatching.

The next analyses focus on how well the local model and the local contingency discrimination model described the run lengths and visit durations. Figure 3 shows for each rat, on log-log coordinates, that mean run lengths and mean visit durations increased as a function of the ratio of the likelihoods of reinforcement for staying and switching. The likelihood of stay reinforcement was the number of stay reinforcers divided by the sum of the durations of visits at the stay lever that earned those reinforcers. Likelihood of reinforcement for switching was the number of reinforcers for switching divided by the sum of the durations of visits at the stay lever that

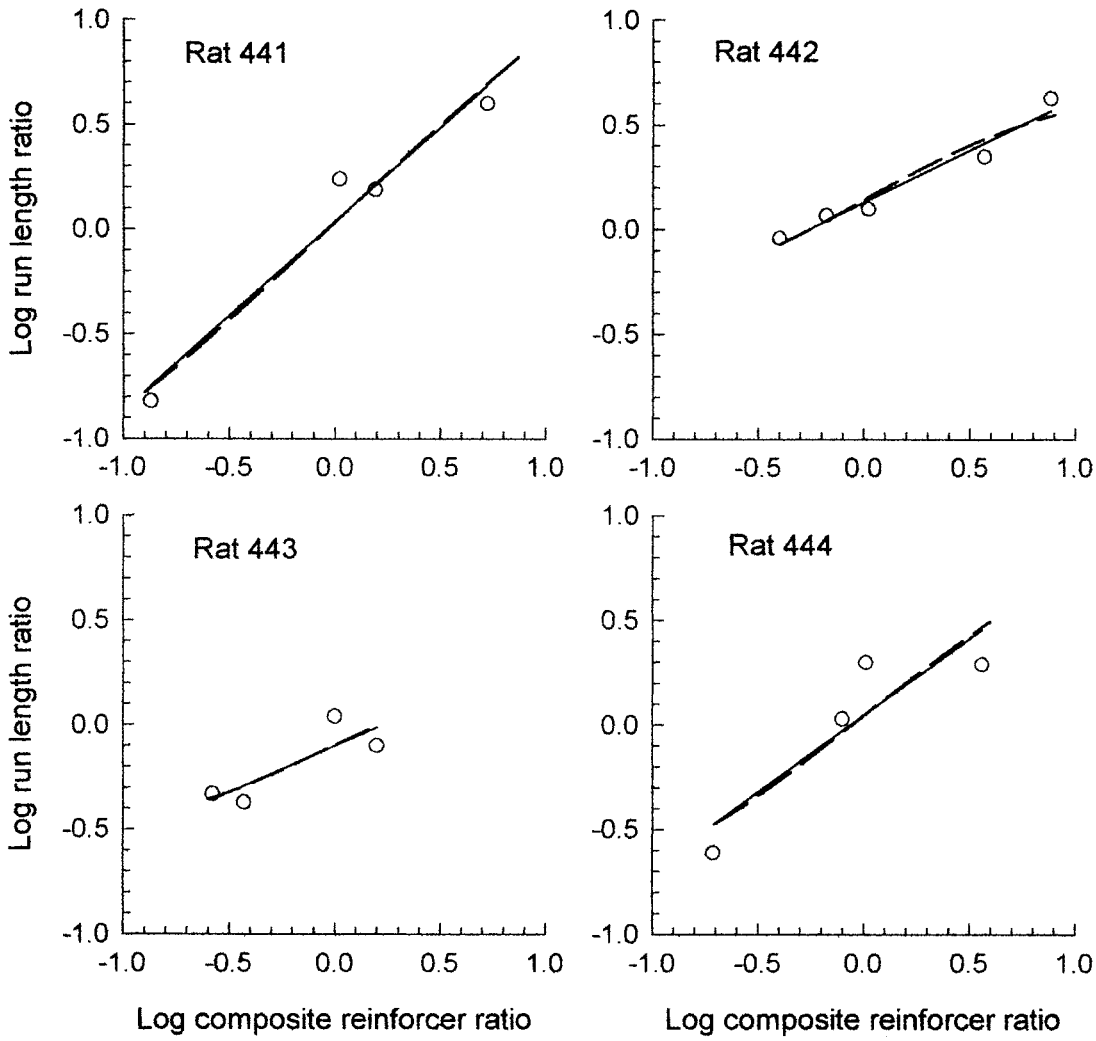

Fig. 1. The log of the composite response ratios as a function of the log of the composite reinforcer ratios. The solid lines represent the best fitting line according to the generalized matching law (Equation 2), and the dashed lines represent the best fitting line according to the local contingency discrimination model (Equation 7).

earned those reinforcers. Two conditions had identical programmed contingencies in which the probability generator was set to .0078 for arranging both stay and switch reinforcers. Including both these data points in the regression necessarily decreases the r^2 of the fitted function unless, of course, the data points are identical. Consequently, Equation 3 was fitted to the data using the mean of these two points. The increasing function is approximately linear for both run lengths and visit durations for all rats, with the exception of run lengths for Rat 442.

Table 3 presents the results of least squares fits of run lengths and visit durations by Equation 3, a power function, which, with the exception of run lengths for Rat 442, are fair to excellent ($r^2 > .60$). The low slope in several conditions makes r^2 a poor estimate of the quality of the fit. However, the standard errors of estimate of the slope and y intercept are moderate, indicating that the fits are good, despite the occasional small r^2. For each rat, the slopes (k_1) for run lengths and visit durations are approximately equal. For 3 rats, the y intercepts (log m_1) are larger for visit duration than for run length, reflecting the generally longer visit durations compared

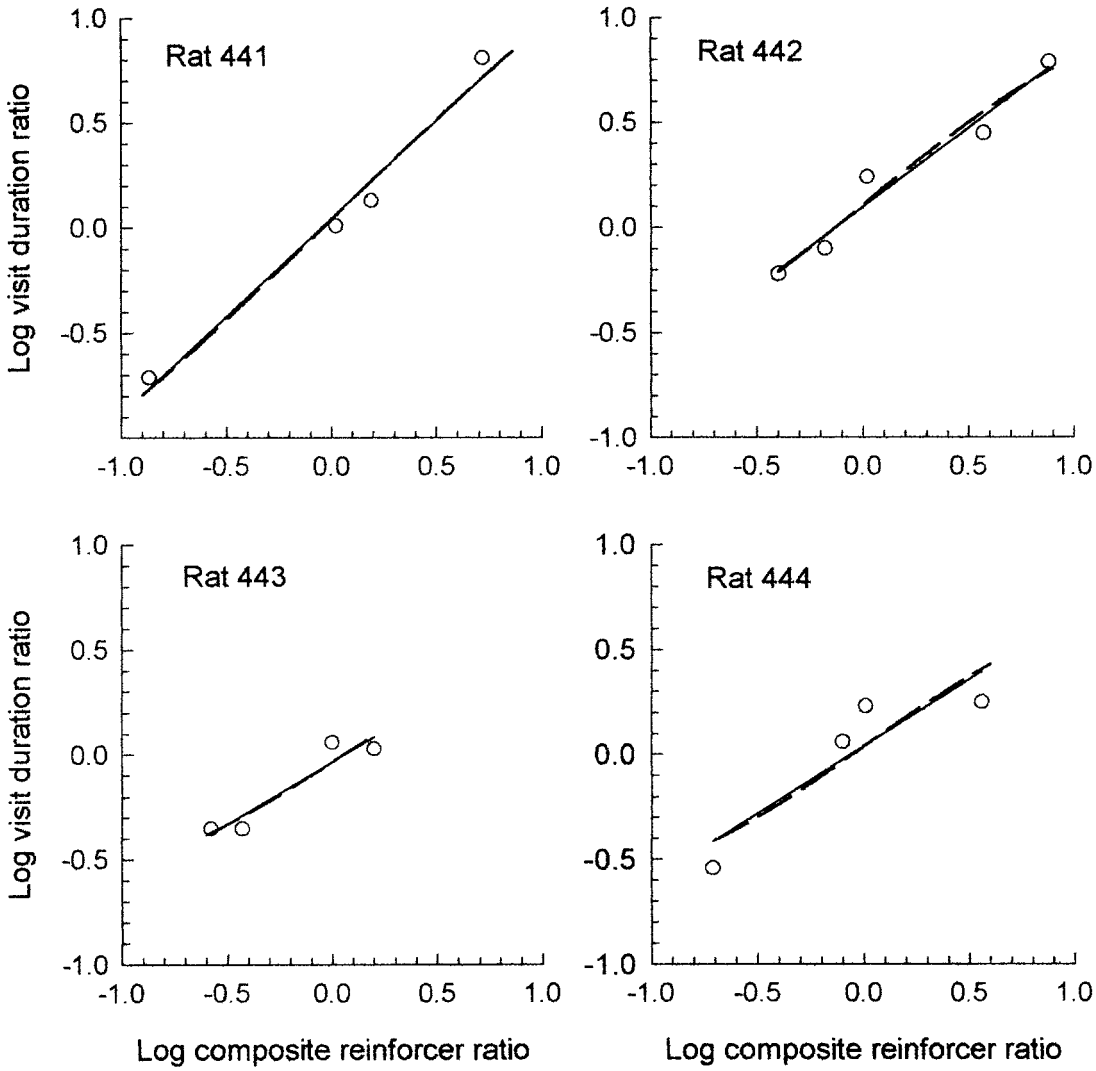

Fig. 2. The log of the composite visit duration ratios as a function of the log of the composite reinforcer ratios. The solid lines represent the best fitting line according to the generalized matching law (Equation 2), and the dashed lines represent the best fitting line according to the local contingency discrimination model (Equation 7).

to run lengths. The opposite was the case for Rat 443.

Figure 4 shows the same data as in Figure 3 but because Equation 6 is expressed linearly, the data are plotted on linear coordinates. The lines are the best fits, by the method of least squares, of the local contingency discrimination model. Table 4 presents the fits to these data for each rat. Equation 6 was fitted to the data using the user-defined equation in SigmaPlot's™ regression routine. The fits for run lengths and visit durations are fair to excellent ($r^2 > .69$). The standard errors of estimate were moderate to small. Inspection of Figures 3 and 4 and Tables 3 and 4 reveals that neither the local model (Equation 3) nor the local contingency discrimination model (Equation 6) is consistently better at fitting the data. The local contingency discrimination model provides a better fit to the most discrepant data, the run-length data of Rat 442.

The next analyses assess the fits of the local model and the local contingency discrimination model to the composite concurrent performances. Figure 5 shows that the log of the

Table 2

Fits by Equation 2, the per-visit version of the generalized matching law, to the log ratio of run lengths and of visit durations as a function of the log composite reinforcer ratios.

Rat	a	SE	log b	SE	df	r^2
Run lengths						
441	0.91	0.14	0.04	0.08	2	.96
442	0.49	0.06	0.13	0.03	3	.96
443	0.45	0.21	−0.10	0.08	2	.69
444	0.74	0.25	0.05	0.12	2	.81
Visit durations						
441	0.94	0.09	0.04	0.05	2	.98
442	0.75	0.09	0.10	0.05	3	.96
443	0.60	0.14	−0.03	0.05	2	.90
444	0.64	0.22	0.04	0.10	2	.82

Table 3

Fits by Equation 3, the local model, to log run lengths and to log visit durations as a function of the log ratio of obtained likelihoods of reinforcement (stay/switch).

Rat	k_1	SE	log m_1	SE	df	r^2
Run lengths						
441	0.46	0.03	1.35	0.02	5	.98
442	0.28	0.14	1.07	0.08	7	.37
443	0.29	0.10	1.51	0.04	5	.61
444	0.37	0.06	1.61	0.03	5	.90
Visit durations						
441	0.40	0.11	1.69	0.06	5	.75
442	0.40	0.05	1.58	0.03	7	.90
443	0.31	0.08	1.38	0.03	5	.77
444	0.32	0.05	1.65	0.03	5	.87

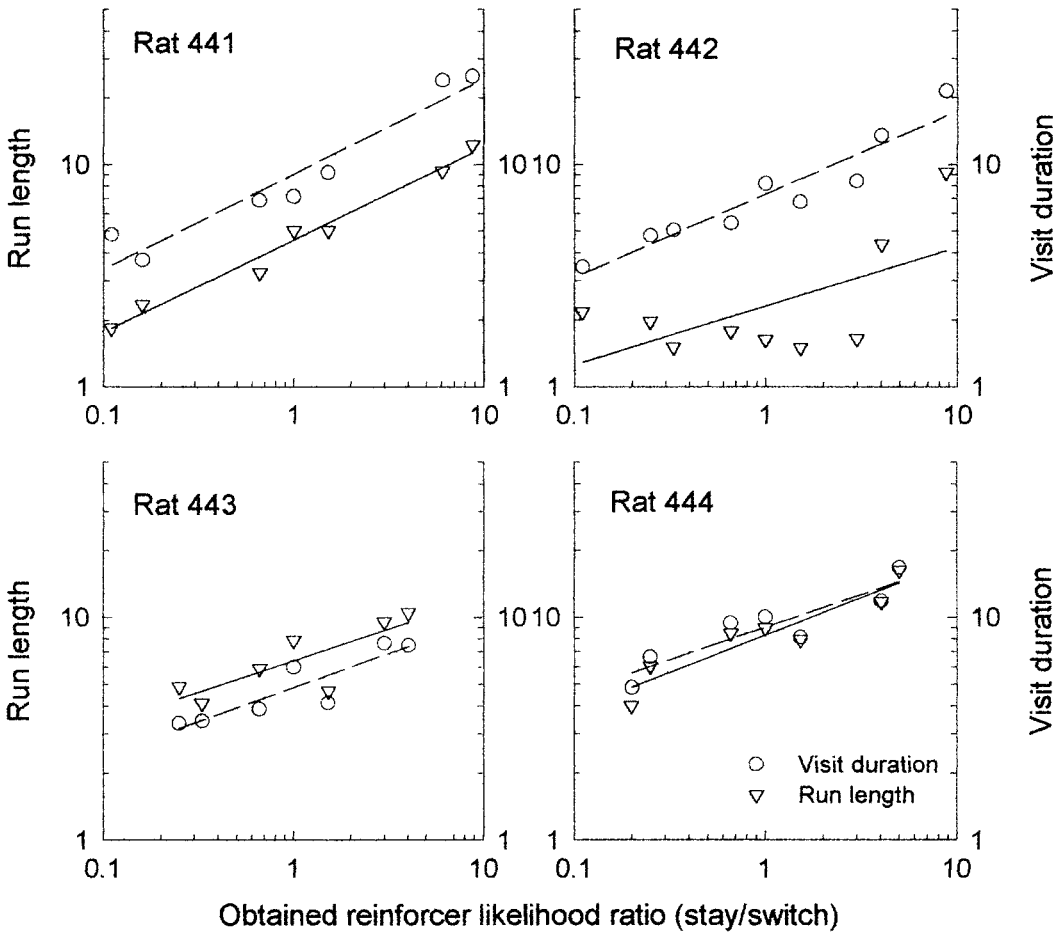

Fig. 3. On log-log coordinates, run lengths (inverted triangles and solid lines) and visit durations (circles and dashed lines) as a function of the obtained likelihood of reinforcement for staying divided by the obtained likelihood of reinforcement for switching. The lines represent the best fitting lines according to the local model (Equation 3).

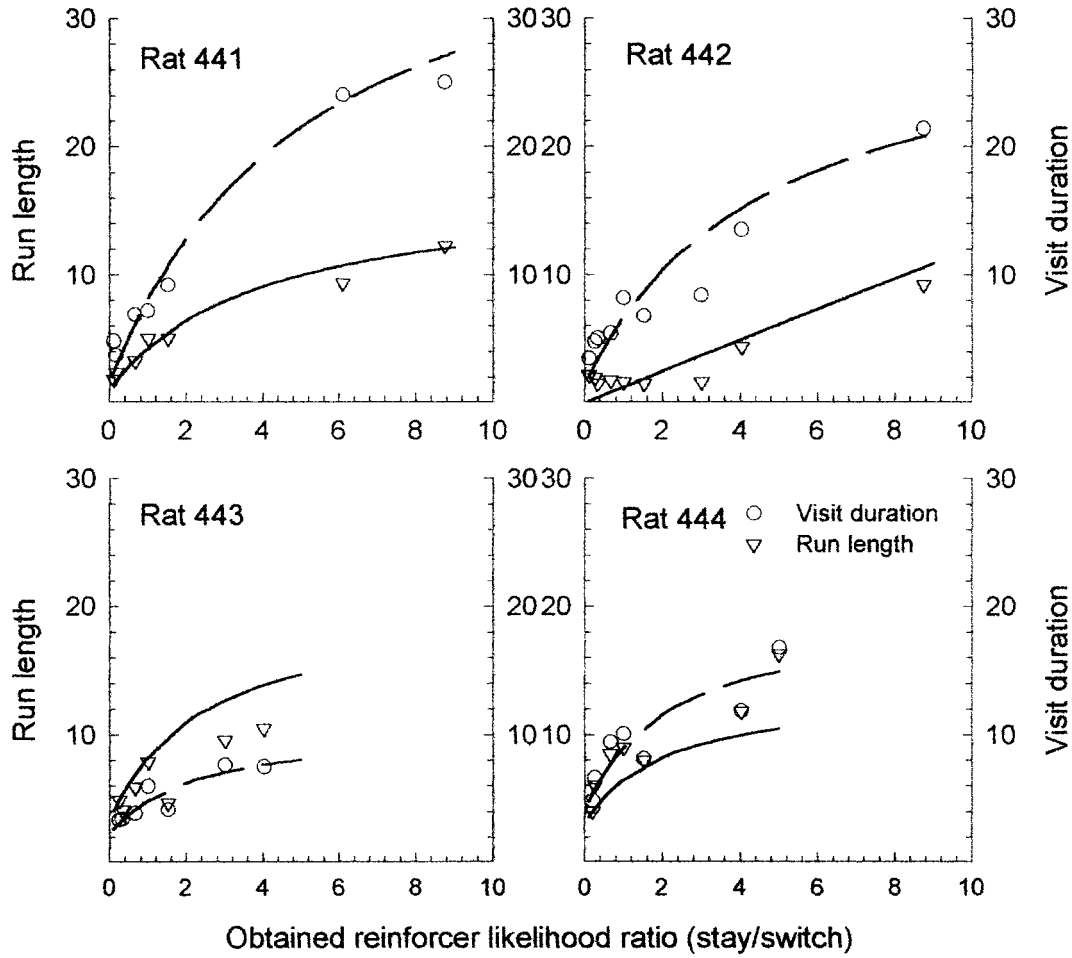

Fig. 4. On linear coordinates, run lengths (inverted triangles and solid lines) and visit durations (circles and dashed lines) as a function of the obtained likelihood of reinforcement for staying divided by the obtained likelihood of reinforcement for switching. The lines represent the best fitting lines according to the local contingency discrimination model (Equation 6).

Table 4

Fits by the local contingency discrimination model, Equation 6, to run lengths and to visit durations as a function of the ratio of the obtained likelihoods of reinforcement (stay/switch).

Rat	p_1	*SE*	n_1	*SE*	*df*	r^2
Run lengths						
441	0.20	0.02	4.30	0.29	5	.98
442	0.00	0.05	1.25	0.40	7	.86
443	0.29	0.03	8.25	0.54	5	.92
444	0.32	0.06	6.45	0.65	5	.69
Visit durations						
441	0.16	0.03	8.16	0.87	5	.97
442	0.18	0.04	6.69	0.80	7	.91
443	0.31	0.05	4.85	0.41	5	.78
444	0.31	0.03	8.96	0.64	5	.87

ratio of run lengths and the log of the ratio of visit durations increased approximately linearly with the log of the ratio of the products of the likelihoods of reinforcement at each alternative. Table 5 presents the results of least squares fits by Equation 4 to these data, which are generally good to excellent ($r^2 >$ .82), with the exception of the run-length data of Rat 443. The standard errors were moderate to small. There was a slight bias, log $m' > 0$, for 3 rats. Within rats, there was no consistent difference in slopes, k', for run lengths and visit durations. Table 5 also shows the slopes, a from fits by the generalized matching law (Table 2) and k_1 from the local model (Table 3). In all eight comparisons, a

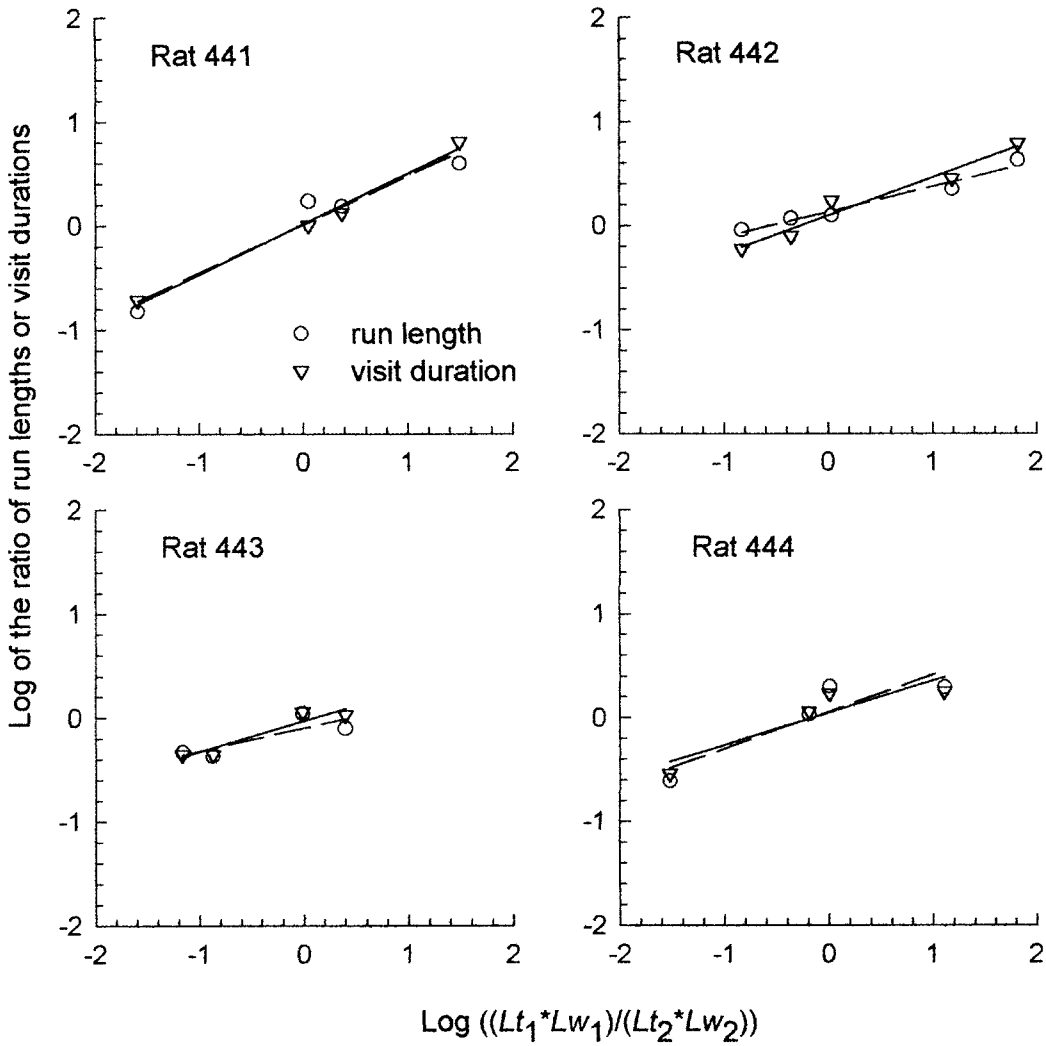

Fig. 5. The log of the ratio of run lengths (inverted triangles and solid lines) and of visit durations (circles and dashed lines) as a function of the log of the ratio of the products of the likelihoods of reinforcement for staying at and switching to Alternative 1 divided by the products of the likelihoods of reinforcement for staying at and switching to Alternative 2. The lines represent the best fitting power functions (Equation 4).

is approximately twice k', and k' approximately equals k_1.

The local contingency discrimination model, Equation 6, produced a model of concurrent performance, Equation 7. The dashed lines in Figures 1 and 2 show fits of Equation 7 to these data. Equation 7 was fitted to these data by the least squares method using the user-defined equation in SigmaPlot's™ regression routine. Table 6 shows these fits, which are all excellent ($r^2 > .89$).

Baum, Schwendiman, and Bell (1999) showed that when plots of run lengths as a function of preference are analyzed separately for the richer and leaner alternatives, the slopes for the two functions are quite different. At the richer alternative (the alternative with the higher rate of reinforcement), the slope is positive and greater than zero, whereas at the leaner alternative, the slope is approximately zero. Figure 6 shows that in the present experiment, the log of the run lengths increased at the richer alternative and decreased or remained constant at the leaner alternative as preference for the richer alternative increased. Because it was not clear

Table 5

Fits by Equation 4, the local model applied to concurrent performance, to the log ratio of run lengths and of visit durations as a function of the log of the ratio of products of obtained reinforcer likelihoods. To facilitate comparisons, a from Table 2 and k_1 from Table 3 are reproduced below.

Rat	k'	*SE*	log m'	*SE*	*df*	r^2	a	k_1
Run lengths								
441	0.47	0.08	0.01	0.09	2	.94	0.91	0.46
442	0.24	0.03	0.13	0.03	3	.96	0.49	0.28
443	0.22	0.11	−0.10	0.08	2	.69	0.45	0.29
444	0.36	0.12	0.05	0.11	2	.82	0.74	0.37
Visit durations								
441	0.49	0.04	0.02	0.04	2	.99	0.99	0.40
442	0.36	0.04	0.10	0.05	3	.96	0.74	0.40
443	0.30	0.07	−0.03	0.05	2	.90	0.60	0.31
444	0.31	0.10	0.05	0.10	2	.83	0.64	0.32

Table 6

Fits by the local contingency discrimination model as applied to concurrent performances, Equation 7, to the per-visit data.

Rat	p'	*SE*	n'	*SE*	*df*	r^2
Run lengths						
441	0.31	0.04	1.35	0.29	2	.94
442	0.27	0.04	0.91	0.25	3	.93
443	0.28	0.06	0.94	0.09	2	.89
444	0.22	0.06	0.96	0.14	2	.97
Visit durations						
441	0.16	0.02	0.89	0.14	2	1.00
442	0.19	0.02	0.76	0.13	3	.99
443	0.25	0.01	1.08	0.02	2	1.00
444	0.27	0.02	0.97	0.06	2	.99

from Baum et al.'s analysis how to handle the data from the composite conditions when the scheduled likelihoods of reinforcement were the same (.0078), these data were omitted from the present analyses.

Figure 7 shows, for each rat, that the efficiency of obtaining stay reinforcers did not change systematically as a function of the ratio of the scheduled likelihoods of stay and switch reinforcement. In contrast, the efficiency of obtaining switch reinforcers increased with the ratio of the scheduled likelihoods of stay and switch reinforcement. The efficiency of obtaining stay reinforcers was calculated by dividing the number of reinforcers obtained for staying by the maximum number of stay reinforcers that could be obtained. The maximum number of stay reinforcers that could be obtained was the total duration at the stay lever, which earned stay reinforcers, divided by the scheduled stay RI (0.5 s divided by probability of an output). Similarly, the efficiency of obtaining switch reinforcers was calculated by dividing the number of reinforcers obtained for switching by the maximum number of switch reinforcers that could be obtained. The maximum number of switch reinforcers that could be obtained was the total duration at the stay lever, which also earned switch reinforcers, divided by the scheduled switch RI. Thus, the efficiency of obtaining switch reinforcers decreased as the relative likelihood of switch reinforcement increased. Some efficiencies are greater than 100%, which probably reflects local fluctuations in the random output generator used in the RI schedule. This is especially likely given the low likelihood of an output and the few total outputs at an alternative, particularly at the extremes of the reinforcer likelihood ratios. The fewer the total reinforcers, the more likely that local fluctuations will produce large deviations from the expected frequency of outputs. Even considering this, there seems to be a surprisingly large number of conditions with efficiency ratios greater than 100%. It is unlikely that these local fluctuations can account for the lack of a relation among stay efficiency and the ratio of the scheduled likelihoods of reinforcement and the direct relation between switch efficiencies and the ratio of the scheduled likelihoods of reinforcement.

DISCUSSION

Composite concurrent performances, obtained by combining paired performances, were described by the generalized matching relation (Figures 1 and 2 and Table 2). These paired performances were functionally unrelated and were obtained from different sessions, sometimes widely separated in time. These results replicate and extend to interval schedules the results of MacDonall (1999), which showed that performance maintained by functionally unrelated pairs of VR schedules can produce composite concurrent VR performances that are described by the generalized matching law.

Neither the composite concurrent VR performances nor the present composite concurrent RI performances can be explained by any

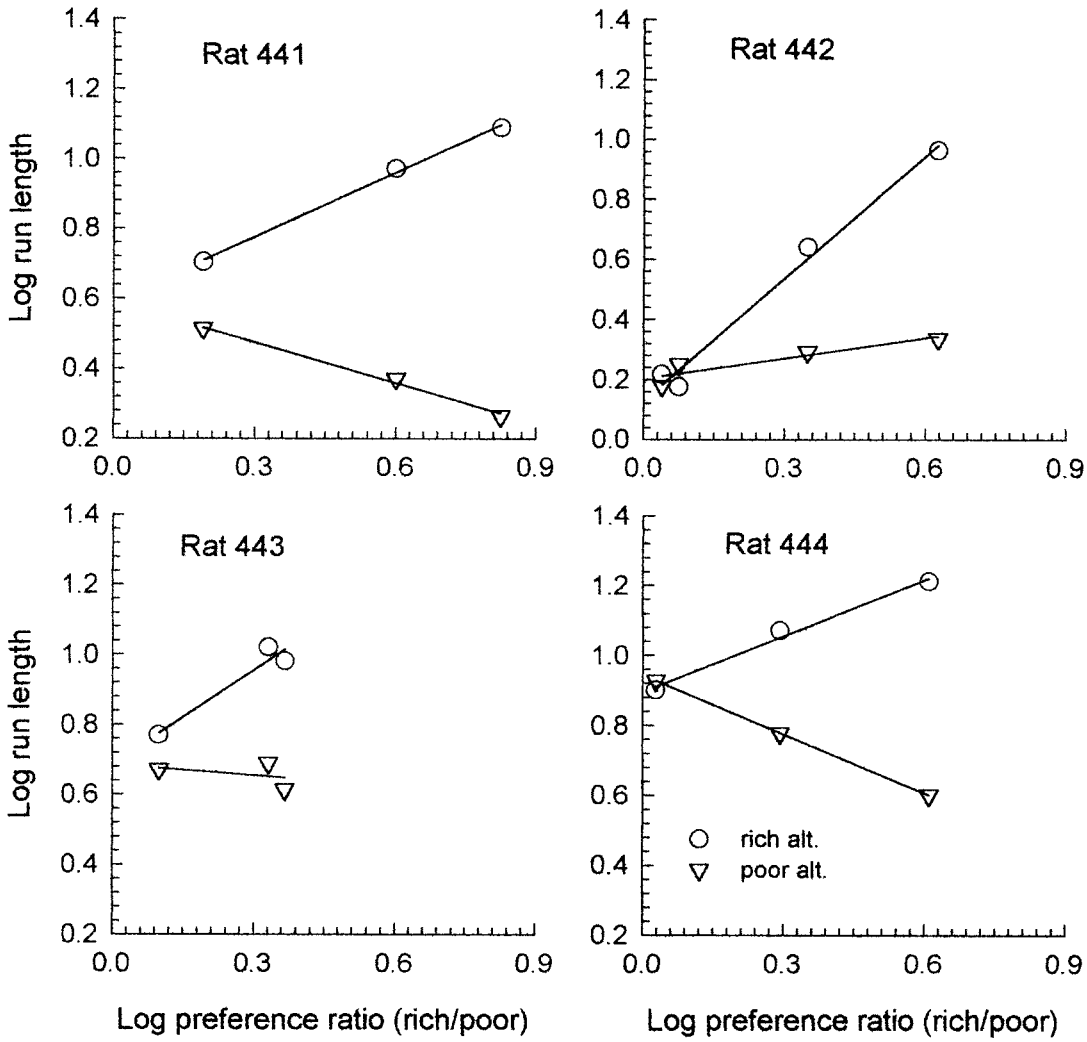

Fig. 6. Run lengths as a function of preference for the richer alternative. The preference for the richer alternative was the ratio of the run lengths at each alternative; the run length for the schedule with the greater per-visit frequency of reinforcement was divided by the run length for the schedule with the smaller per-visit frequency of reinforcement.

theory of concurrent performance that, either explicitly or implicitly, requires comparisons among the alternatives. This is because in the composite performances the alternatives do not alternate, making comparisons impossible. Scalar expectancy theory (Gibbon, Church, Fairhurst, & Kacelnik, 1988) explicitly proposes that matching results from comparisons among the alternatives. Organisms store interreinforcement intervals at each alternative. Then they randomly sample one interval from each distribution of interreinforcement intervals. They compare the two randomly sampled intervals and respond at the alternative associated with the smaller interval. In composite concurrent performances, this comparison appears to be impossible because the two alternatives do not alternate within a session. A simple modification may allow scalar expectancy theory to account for the present data as well as data from concurrent procedures. For example, it may be proposed that the interreinforcement intervals sampled from memory are the distribution of stay interreinforcement intervals and the distribution of switch interreinforcement intervals associated with the current alternative. However, a difficulty with this approach is that the RI sched-

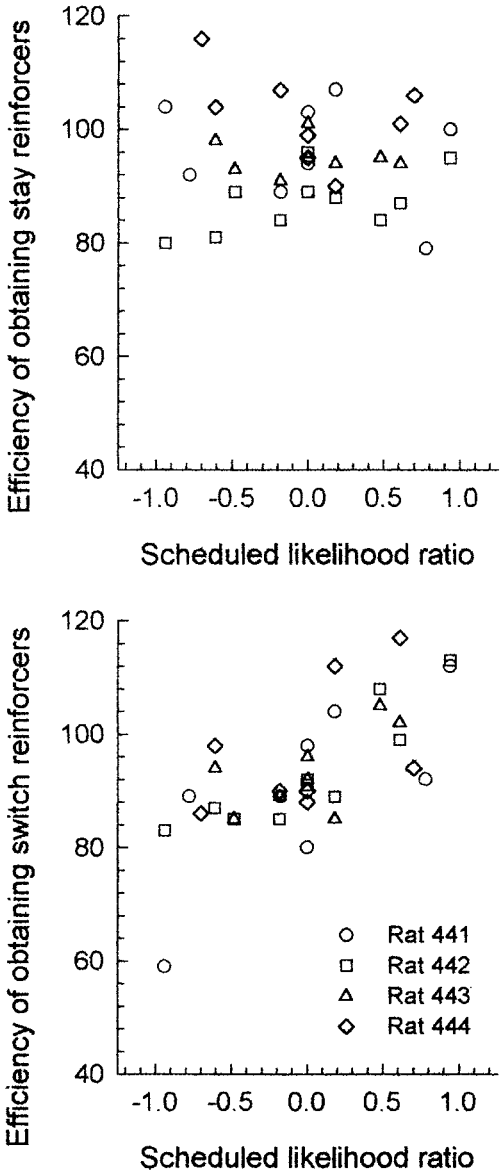

Fig. 7. The efficiency of obtaining stay reinforcers (top panel) and switch reinforcers (bottom panel) as a function of the ratio of the scheduled likelihoods of stay divided by switch reinforcement.

ules stopped following a switch response and resumed with the next stay response. Whether animals can differentiate when the RI schedules are operating, as apparently required for scalar expectancy theory to account for the present data, is an empirical question. Finally, for this modification to apply to concurrent procedures, two interreinforcement intervals need to be stored and then sampled for each alternative. This results in a total of four stored distributions of interreinforcement intervals.

Matching as a fundamental process (Herrnstein, 1970), melioration (Vaughan, 1981), and contingency discrimination, as originally proposed (Davison & Jenkins, 1985), all implicitly require comparisons among the alternatives. For these theories of matching, comparisons among the alternatives are necessary, although they are not explicitly discussed. Herrnstein suggested that organisms behave so as to produce behavior allocations that equal reinforcer allocations. Certainly, some sort of comparison of behavior and reinforcer allocation is implicit in this view. Melioration begins to suggest the comparison implicit in matching. In melioration, the local rates of reinforcement are thought to be crucial. Animals switch to the alternative with the higher local rate of reinforcement, which implies a comparison of local rates of reinforcement at the alternatives. Contingency discrimination implies a comparison of perceived reinforcer allocation and behavior allocation. Although the generalized matching law described the composite concurrent performances, because the alternatives did not alternate and usually did not succeed each other, any of these comparisons appear to be impossible. In addition, to the extent that the variables influencing the composite concurrent interval performances accurately reflect the variables influencing standard concurrent interval performances (cf. Herrnstein, 1961), concurrent performances could not be understood by any theory positing, explicitly or implicitly, comparisons among the alternatives.

The results of the present experiment help us to understand how concurrent performance occurs. The performances at the alternatives were independent of one another, which implies that models of concurrent performance need to focus on the contingencies and performances at the alternatives. That is, one needs to develop a model of performance based on the contingencies at each of the alternatives. One approach to modeling concurrent performance is the local version of the contingency discrimination model (Equation 6), which fitted the performances at the alternatives, that is, run lengths, visit durations, and stay and switch reinforcers (Figure 4 and Table 4). Note that this model

combines the essential features of the contingency discrimination model (Davison & Jenkins, 1985) and the local model (MacDonall, 1999). The local contingency discrimination model, as applied to concurrent performance (Equation 7), fitted the composite results (Figures 1 and 2 and Table 6).

Another approach to modeling concurrent performance is the local model (Equation 3), which fitted the run lengths, visit durations, and stay and switch reinforcers per visit. Run lengths and visit durations were power functions of the ratio of the likelihoods of reinforcement. From these independently acquired run lengths and visit durations, composite concurrent performances were created, and the local model, as applied to concurrent performance (Equation 4) fitted these composite performances. These results replicate and extend to interval-based stay and switch schedules, which are commonly used in concurrent procedures, those obtained using VR as the stay and switch schedules (MacDonall, 1999). A comparison of the fits by the local model as applied to concurrent performance and the local contingency discrimination model as applied to concurrent performance (Equations 4 and 7, respectively) show that r^2 was larger for four of the fits by the local contingency discrimination model (see Tables 5 and 6).

In evaluating a model, factors in addition to the goodness of fit of the model to the data are considered. One factor is the validity of the assumptions. For example, the local contingency discrimination model, as applied to concurrent performance, assumes that reinforcer allocation is perceived equally well at each alternative, that is, the model assumes $p_1 = p'$. However, the results of the present experiment do not support this assumption. Inspection of Tables 4 and 6 shows that $p_1 = p'$ in only three of the eight comparisons. The local model assumes that behavioral sensitivity to reinforcer allocation is equal at the alternatives, $k_1 = k'$. This equality was supported by the present data (Table 5) and by MacDonall (1999).

MacDonall (1999) discussed the relation between the generalized matching law (Equation 1) and the local model as applied to concurrent performances (Equation 4). He demonstrated that both equations can be derived from Equation 3, the local model, and that the slope from the generalized matching law, a, should be twice k', the slope from the local model as applied to concurrent performance (Equation 4). His results showed that, in fact, a was approximately twice k' for each rat. This relation was also found for each rat in the present experiment (Table 5). Because the generalized matching law can be derived from the local model, this suggests that the orderly performances at the alternatives are not by-products of generalized matching; rather, generalized matching is a by-product of orderly performances at the alternatives. The view that the local model may be more fundamental was also based on the decomposition of concurrent schedules into two pairs of schedules, and the creation of composite concurrent performances from functionally unrelated performances maintained by just one pair of schedules. Because those composite performances were obtained using one pair of VR schedules, and because there is only one published demonstration of nonexclusive matching using VR schedules (MacDonall, 1988), some may be concerned about extending this analysis to concurrent interval schedules. The present results were obtained using functionally unrelated pairs of interval schedules, indicating that the argument that the local model may be the more fundamental relation also applies to concurrent interval schedules (MacDonall, 1999).

The results of the present rich and lean analysis (Figure 6) partially replicated the results of a similar analysis reported by Baum et al. (1999). Their experiment and the present experiment found that run length at the richer alternative increased as preference for the richer alternative increased. However, at the leaner alternative, where Baum et al. found that run lengths remained approximately constant at one as preference for the richer alternative increased, the present experiment found clear changes: decreases for 2 rats and an increase for 1 rat. Several procedural factors may account for this difference. Baum et al. used a wider range of reinforcer ratios between the richer and leaner alternatives, approximately 5 $\log_{10}$ units, whereas the range in the present experiment was up to 2 $\log_{10}$ units. The greater range in their experiment means the likelihood of reinforcement at the lean alternative was low in many conditions, which may have made it eas-

ier to see that run lengths were approximately equal to one. An additional factor is the difference in the number of sessions per condition in each experiment. Their subjects experienced many more sessions at each condition, ranging from 24 to 147 with a mean of 57 and a median of 50. In the present experiment the number of sessions per condition ranged from 10 to 36 with a mean of 16.5 and a median of 15. If the current experiment were continued for additional sessions in each condition, the results might turn out to be more similar to their results.

The present results and those of MacDonall (1999) are consistent with momentary maximizing accounts of concurrent performance (Shimp, 1966). Likelihoods of reinforcement (in the present experiment) and probabilities of reinforcement (MacDonall, 1999) controlled behavior. However, MacDonall (1999) used a set of ratios obtained by the method of Fleshler and Hoffman (1962), which produces only an approximation to a random probability of reinforcement, which momentary maximizing assumes. The present experiment used RI schedules. Momentary maximizing predicts a pattern consisting of a run at the richer alternative followed by one response at the leaner alternative. The changes in run length at the richer alternative were consistent with momentary maximizing, but the changes in run lengths at the leaner alternative were not. Interestingly, Baum et al. (1999) found that run lengths increased at the richer alternative and remained approximately constant, equaling one, at the leaner alternative, a finding consistent with momentary maximizing. Although others have provided data to challenge momentary maximizing (Heyman, 1979; Nevin, 1969, 1979; Silberberg, Hamilton, Ziriax & Casey, 1978), MacDonall (1999) discussed how those results could be interpreted without challenging momentary maximizing. Essentially, he argued that the large degree of variability in those experiments obscures the molecular relations. The large variability may require means, or other measures of central tendency, rather than sequential dependencies to reveal the molecular relations.

In summary, decomposing concurrent schedules into two pairs of stay and switch schedules results in procedures that inform the controlling relations in concurrent procedures. Each pair of schedules controls a performance that is functionally unrelated to the other pair of schedules and the performances they control. Two models, the local model (Equation 3) and the local contingency discrimination model (Equation 6), fitted the functionally unrelated performances, and each yielded a model that fitted the composite concurrent performances. The functionally unrelated performances at the alternatives and composite concurrent performances that were described by the generalized matching law challenge theories of concurrent performance, such as melioration, scalar expectancy theory, and the original version of the contingency discrimination model (Equation 5), that require comparisons among the alternatives.

REFERENCES

Baum, W. M. (1974). On two types of deviations from the matching law: Bias and undermatching. *Journal of the Experimental Analysis of Behavior, 22,* 231–242.

Baum, W. M., Schwendiman, J. W., & Bell, K. E. (1999). Choice, contingency discrimination, and foraging theory. *Journal of the Experimental Analysis of Behavior, 71,* 355–373.

Davison, M. C., & Jenkins, P. E. (1985). Stimulus discriminability, contingency discriminability, and schedule performance. *Animal Learning & Behavior, 13,* 77–84.

Davison, M. C., & Jones, B. M. (1995). A quantitative analysis of extreme choice. *Journal of the Experimental Analysis of Behavior, 64,* 147–162.

Fleshler, M., & Hoffman, H. S. (1962). A progression for generating variable-interval schedules. *Journal of the Experimental Analysis of Behavior, 5,* 529–530.

Gibbon, J., Church, R. M., Fairhurst, S., & Kacelnik, A. (1988). Scalar expectancy theory and choice between delayed rewards. *Psychological Review, 95,* 102–114.

Green, L., Rachlin, H., & Hanson, J. (1983). Matching and maximizing with concurrent ratio-interval schedules. *Journal of the Experimental Analysis of Behavior, 40,* 217–224.

Herrnstein, R. J. (1961). Relative and absolute strength of response as a function of frequency of reinforcement. *Journal of the Experimental Analysis of Behavior, 4,* 267–272.

Herrnstein, R. J. (1970). On the law of effect. *Journal of the Experimental Analysis of Behavior, 13,* 243–266.

Heyman, G. M. (1979). A Markov model description of changeover probabilities on concurrent variable-interval schedules. *Journal of the Experimental Analysis of Behavior, 31,* 41–51.

Houston, A. I., & McNamara, J. (1981). How to maximize reward rate on two variable interval paradigms. *Journal of the Experimental Analysis of Behavior, 35,* 367–396.

MacDonall, J. S. (1988). Concurrent variable-ratio schedules: Implications for the generalized matching law. *Journal of the Experimental Analysis of Behavior, 50,* 55–64.

MacDonall, J. S. (1998). Run length, visit duration, and reinforcers per visit in concurrent performance. *Journal of the Experimental Analysis of Behavior, 69,* 275–293.

MacDonall, J. S. (1999). A local model of concurrent performance. *Journal of the Experimental Analysis of Behavior, 71,* 57–74.

Nevin, J. A. (1969). Interval reinforcement of choice behavior in discrete trials. *Journal of the Experimental Analysis of Behavior, 12,* 875–885.

Nevin, J. A. (1979). Overall matching versus momentary maximizing: Nevin (1969) revisited. *Journal of Experimental Psychology: Animal Behavior Processes, 5,* 300–306.

Shimp, C. P. (1966). Probablistically reinforced choice behavior in pigeons. *Journal of the Experimental Analysis of Behavior, 9,* 443–455.

Silberberg, A., Hamilton, B., Ziriax, J. M., & Casey, J. (1978). The structure of choice. *Journal of Experimental Psychology: Animal Behavior Processes, 4,* 368–398.

Vaughan, W., Jr. (1981). Melioration, matching, and maximizing. *Journal of the Experimental Analysis of Behavior, 36,* 141–149.

Received December 20, 1999
Final acceptance June 1, 2000

APPENDIX

To develop a model of concurrent performance based on Equation 3, take the ratio of the power functions at each alternative, which yields

$$\log\frac{\left(\frac{B_1}{C_2}\right)}{\left(\frac{B_2}{C_1}\right)} = \frac{k_1\log\left(\frac{Lt_1}{Lw_2}\right) + \log\ m_1}{k_2\log\left(\frac{Lt_2}{Lw_1}\right) + \log\ m_2}. \quad \text{(A1)}$$

When rearranged, this becomes

$$\log\frac{\left(\frac{B_1}{C_2}\right)}{\left(\frac{B_2}{C_1}\right)} = k_1\log\left(\frac{Lt_1}{Lw_2}\right) + k_2\log\left(\frac{Lw_1}{Lt_2}\right) + \log\left(\frac{m_1}{m_2}\right). \quad \text{(A2)}$$

Letting $m' = m_1/m_2$, assuming $k_1 = k_2$, then substituting k' (MacDonall, 1999) and rearranging results in

$$\log\left[\frac{\left(\frac{B_1}{C_2}\right)}{\left(\frac{B_2}{C_1}\right)}\right] = k'\ \log\left(\frac{Lt_1\cdot Lw_1}{Lt_2\cdot Lw_2}\right) + \log\ m'. \quad \text{(A3)}$$