and substitute into it from (16) we obtain by virtue of (10) and (12b)

$$
\begin{aligned}
{}^*\Gamma^{\nu}_{\lambda\mu} &= \Gamma^{\nu}_{\lambda\mu} - S_{\lambda\mu}{}^{\nu} + 2h^{\alpha\nu} K_{\xi\eta\zeta}\, k_{(\mu}{}^{\beta}\, T^{\xi\,\eta\zeta}_{\lambda)\alpha\beta} - i(S_{\lambda\mu}{}^{\nu} - h^{\alpha\nu} K_{\xi\eta\zeta}\, T^{\xi\,\eta\zeta}_{\lambda\mu\alpha}) \\
&= \Gamma^{\nu}_{\lambda\mu} - (i+1)\, S_{\lambda\mu}{}^{\nu} + h^{\alpha\nu} K_{\xi\eta\zeta}\, [2k_{(\mu}{}^{\beta}\, T^{\xi\,\eta\zeta}_{\lambda)\alpha\beta} + i\, T^{\xi\eta\zeta}_{\lambda\mu\alpha}].
\end{aligned}
$$

This formula establishes the relationship between the connections ${}^*\Gamma^{\nu}_{\lambda\mu}$ and $\Gamma^{\nu}_{\lambda\mu}$.

* Prepared under Army Contract DA 33-008 ORD-467.

[1] Einstein, Albert, "A Generalization of the Relativistic Theory of Gravitation," *Ann. Math.* (2), **46**, 578–584 (1945) and "A Generalized Theory of Gravitation," *Rev. Modern Physics*, **20**, 35–39 (1948). Einstein, A., and Straus, E. G., "A Generalization of the Relativistic Theory of Gravitation, II," *Ann. Math.* (2), **47**, 731–741 (1946).

[2] Einstein, A., *The Meaning of Relativity*, Princeton University Press, 1950, Appendix II.

[3] The existence of $\left\{ {\nu \atop \lambda\mu} \right\}$ is possible if and only if the determinant h of $h_{\lambda\mu}$ is different from zero.

[4] The indices are raised and lowered by means of $h^{\lambda\nu}$ and $h_{\lambda\mu}$.

[5] Hlavatý, V., "The Elementary Basic Principles of the Unified Theory of Relativity. B," *J. Rat. Mech. Anal.* (1), **2**, 1–52 (1953).

[6] Hlavatý, V., and Säenz, A. W., "Uniqueness Theorems in the Unified Theory of Relativity," to be published in the *J. Rat. Mech. Anal.*

[7] These conditions may be formulated in a similar way as it has been done for the system (12*b*) in the paper mentioned in Ref. 5.

ON THE CONTRACTION OF GROUPS AND THEIR REPRESENTATIONS

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Communicated April 21, 1953

Introduction.—Classical mechanics is a limiting case of relativistic mechanics. Hence the group of the former, the Galilei group, must be in some sense a limiting case of the relativistic mechanics' group, the representations of the former must be limiting cases of the latter's representations. There are other examples for similar relations between groups. Thus, the inhomogeneous Lorentz group must be, in the same sense, a limiting case of the de Sitter groups. The purpose of the present note is to investigate, in some generality, in which sense groups can be limiting cases of other groups (Section I), and how their representations can be obtained from the representations of the groups of which they appear as limits (Section II). Section III deals briefly with the transition from inhomogeneous Lorentz group to Galilei group. It shows in which way the representation up to a factor of the Galilei group, embodied in

the Schrödinger equation, appears as a limit of a representation of the inhomogeneous Lorentz group and also gives the reason why no physical interpretation is possible for the real representations of that group.

I. CONTRACTION OF GROUPS

Let us consider an arbitrary Lie group with n parameters a^i and infinitesimal operators I_i. These shall be given, as usual by

$$I_i = \lim_{h \to 0} \frac{g(he_i) - g(0)}{h} \tag{1}$$

where 0 are the parameters of the unit element and e_i differs from 0 by a unit increase of a_i. The I_j are skew hermitean matrices if the group consists of unitary matrices; instead of them one often uses the hermitean quantities iI_j. However, all our equations remain somewhat simpler if expressed in terms of the I_j. The structure constants C are defined by

$$[I_i, I_j] = \sum_k C_{ij}^k I_k \tag{2}$$

If we subject the I_i to a linear homogeneous non-singular transformation, the C will be replaced by other constants. These are obtained from the C by contragradient transformations of its upper and lower indices. However, such a transformation has, naturally, no effect on the structure of the group. Let us denote the transformation in question by

$$J_\nu = \sum_i I_i U^i{}_\nu. \tag{3}$$

It corresponds to the transformation

$$a^i = \sum_\kappa U^i{}_\kappa b^\kappa \tag{3a}$$

according to which the J are obtained by the same equation (1) as the I, except that the e_i have to be replaced in it by a similar quantity, defined with respect to the b.

The above transformation may lead to a new group only if the matrix U of (3) is singular. We shall call the operation of obtaining a new group by a singular transformation of the infinitesimal elements of the old group a *contraction* of the latter. The reason for this term will become clear below. The singular matrix will be a limiting case of a non-singular matrix. The latter will depend linearly on a parameter ϵ which will tend to zero:

$$U^i{}_\nu = u^i{}_\nu + \epsilon w^i{}_\nu. \tag{4}$$

For $0 < \epsilon < \epsilon_0$ the determinant of (4) is different from zero, it vanishes for $\epsilon = 0$.

We shall transform (4) into a normal form by a non-singular and ϵ

independent transformation of both the J_ν and I_i. If the matrices of the corresponding transformations are denoted by α and β, respectively, u and w will be replaced by $\beta u \alpha^{-1}$ and $\beta w \alpha^{-1}$. It is possible, by such a transformation, to give u and w the form

$$u = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix} \qquad w = \begin{Vmatrix} v & 0 \\ 0 & 1 \end{Vmatrix}. \tag{5}$$

The number of rows and columns in the unit matrix in u, and in v, is equal to the rank r of u. It is advantageous to label the transformed J and I with a pair of indices, the first referring to the subdivision of u given in (5), the second specifying the various J and I within that subdivision. Hence, (3) assumes the form

$$\begin{aligned} J_{1\nu} &= I_{1\nu} + \epsilon \sum_{\mu=1}^{r} v_{\mu\nu} I_{1\mu} \qquad (\nu = 1, 2, \ldots, r) \\ J_{2\nu} &= \epsilon I_{2\nu}. \qquad (\nu = 1, 2, \ldots, n - r) \end{aligned} \tag{6}$$

The corresponding transformation of the group parameters is

$$\begin{aligned} a_{1\nu} &= b_{1\nu} + \epsilon \sum_{\mu=1}^{r} v_{\nu\mu} b_{1\mu} \qquad (\nu = 1, 2, \ldots, r) \\ a_{2\nu} &= \epsilon b_{2\nu}. \qquad (\nu = 1, 2, \ldots, n - r) \end{aligned} \tag{6a}$$

It is well to remember that the parameters a lead to the infinitesimal elements I, the parameters b to the J. The last equation shows that a given set of parameters b correspond, with decreasing ϵ, to smaller and smaller values of the parameters a_2. In the limit $\epsilon = 0$ (if such a limit exists), one will have contracted the whole group to an infinitesimally small neighborhood of the group defined by the $a_{1\nu}$ alone. This justifies the name given to the process considered.

The transformation of the infinitesimal elements which we carried out also changes the structure constants and we shall write for (2)

$$[I_{\alpha\nu}, I_{\beta\mu}] = \sum_{\kappa=1}^{r} C_{\alpha\nu,\beta\mu}{}^{1\kappa} I_{1\kappa} + \sum_{\kappa=1}^{n-r} C_{\alpha\nu,\beta\mu}{}^{2\kappa} I_{2\kappa} \tag{7}$$

wherein α and β can assume the values 1 and 2. This gives for

$$\begin{aligned} [J_{1\nu}, J_{1\mu}] &= [I_{1\nu}, I_{1\mu}] + \epsilon \sum (v_{\nu\nu'}\delta_{\mu\mu'} + \delta_{\nu\nu'} v_{\mu\mu'} + \epsilon v_{\nu\nu'} v_{\mu\mu'}) [I_{1\nu'}, I_{1\mu'}] \\ &= \sum_{\kappa} C_{1\nu,1\mu}{}^{1\kappa} J_{1\kappa} + \frac{1}{\epsilon} \sum_{\kappa} C_{1\nu,1\mu}{}^{2\kappa} J_{2\kappa} + 0(1). \end{aligned}$$

Hence, if the commutator of $J_{1\nu}$ and $J_{1\mu}$ is to converge, as $\epsilon \to 0$, to a linear combination of the J, the structure constants

$$C_{1\nu,\,1\mu}{}^{2\kappa} = 0 \tag{8}$$

i.e., the $I_{1\nu}$ must span a subgroup. On the other hand, if this happens to be the case, the structure constants will converge to definite values $c_{\alpha\nu,\,\beta\mu}{}^{\gamma\kappa}$ as $\epsilon \to 0$

$$\begin{aligned} c_{1\nu,\,1\mu}{}^{1\kappa} &= C_{1\nu,\,1\mu}{}^{1\kappa} & c_{1\nu,\,1\mu}{}^{2\kappa} &= C_{1\nu,\,1\mu}{}^{2\kappa} = 0 \\ c_{1\nu,\,2\mu}{}^{1\kappa} &= 0 & c_{1\nu,\,2\mu}{}^{2\kappa} &= C_{1\nu,\,2\mu}{}^{2\kappa} \\ & & c_{2\nu,\,2\mu}{}^{1\kappa} &= c_{2\nu,\,2\mu}{}^{2\kappa} = 0. \end{aligned} \tag{9}$$

These structure constants satisfy Jacobi's identities since the structure constants for the J do this for non-vanishing ϵ. We shall say that the above operation is a contraction of the group with respect to the infinitesimal elements $I_{1\nu}$ or that the infinitesimal elements $I_{2\mu}$ are contracted. We then have, from (9).

THEOREM 1. *Every Lie group can be contracted with respect to any of its continuous subgroups and only with respect to these.* The subgroup with respect to which the contraction is undertaken will be called S. *The contracted infinitesimal elements form an abelian invariant subgroup of the contracted group. The subgroup S with respect to which the contraction was undertaken is isomorphic with the factor group of this invariant subgroup. Conversely, the existence of an abelian invariant subgroup and the possibility to choose from each of its cosets an element so that these form a subgroup S, is a necessary condition for the possibility to obtain the group from another group by contraction.*

It is easy to visualize now the effect of the contraction on the whole group. The subgroup S with respect to which the contraction is undertaken remains unchanged and it is advantageous to choose the group parameters in such a way that $a_{2\nu} = 0$ throughout S. Then (6a) can be replaced by

$$a_{1\nu} = b_{1\nu} \qquad a_{2\nu} = \epsilon b_{2\nu} \tag{6b}$$

and this can be assumed to be valid throughout the whole group, not only in the neighborhood of the unit element. As ϵ decreases, a fixed range of the parameter b will describe an increasingly small surrounding of S. As ϵ tends to 0, the range of the $b_{2\nu}$ will become infinite and describe only those group elements which differ infinitesimally from the elements of S. The elements which are in the neighborhood of the unit element of the original group but have finite parameters $b_{2\nu}$ will commute and form the aforementioned commutative invariant subgroup. Naturally, the elements of this invariant subgroup will not commute, in general, with the elements of the subgroup S: the change of the parameters $a_{2\nu} = \epsilon b_{2\nu}$ will be, upon transformation by finite elements of S, of the same order of magnitude as these parameters themselves. Naturally, the convergence

of the original group toward the contracted group is a typically non-uniform convergence.

Every Lie group can be contracted with respect to any of its one parametric subgroups. If the three dimensional rotation group is contracted in this way, one obtains the Euclidean group for two dimensions. Contraction of the homogeneous Lorentz group with respect to the subgroup which leaves the time coordinate invariant yields the homogeneous Galilei group, contraction of the inhomogeneous Lorentz group with respect to the group generated by spatial rotations and time displacements yields the full Galilei group. Contraction of the de Sitter groups yields the inhomogeneous Lorentz group. It should be remarked, finally, that if a group, obtained by contraction of another group with respect to the subgroup S, is contracted again with respect to S, the second contraction remains without effect.

The above considerations show a certain similarity with those of I. E. Segal.[1] However, Segal's considerations are more general than ours as he considers a sequence of Lie groups the structure constants of which converge toward the structure constants of a non-isomorphic group. In the above, we have considered only one Lie group but have introduced a sequence of coordinate systems therein and investigated the limiting case of these coordinate systems becoming singular. As a result of our problem being more restricted, we could arrive at more specific results.

II. Contraction of Representations

If one applies the transformation (6) to the infinitesimal elements of a representation of the group to be contracted, and lets ϵ tend to zero, the $J_{2\nu}$ will also tend to zero. The representation will become isomorphic to the representation of the subgroup S, i.e., will be a representation of the factor group of the invariant subgroup. In order to obtain a faithful representation, one must *either subject the $J_{2\nu}$ to an ϵ dependent transformation, or consider the $J_{2\nu}$ which correspond to different representations*, e.g., go to higher and higher dimensional representations as ϵ decreases. We shall give examples for both procedures.

(*a*) *Representations of the Contracted Group by Means of ϵ Dependent Transformations.*—The first procedure is applicable only if the infinitesimal elements are not bounded operators, i.e., as far as irreducible representations are concerned, only if the group is not compact. The simplest non-compact non-commutative group is that of the linear transformations $x' = e^{\alpha}x + \beta$. The general group element is $O_{\alpha,\beta} = T(\beta)R_{\alpha}$ with the group relations $T(\beta)T(\beta') = T(\beta + \beta')$; $R_{\alpha}R_{\alpha'} = R_{\alpha+\alpha'}$ and $R_{\alpha}T(\beta) = T(e^{\alpha}\beta)R_{\alpha}$. The only faithful irreducible unitary representation of this group can be given in the Hilbert space of square integrable functions of $0 < x < \infty$:

$$R_\alpha\psi(x) = e^{(1/2)\alpha}\psi(e^\alpha x) \qquad T_\beta\psi(x) = e^{i\beta x}\psi(x). \tag{10}$$

The infinitesimal operators are

$$I_\alpha = I_1 = {}^1/_2 + x\frac{d}{dx} \qquad I_\beta = I_2 = ix \tag{10a}$$

with the structure relation

$$[I_1, I_2] = I_2. \tag{10b}$$

Contraction with respect to the group of transformations $x' = e^{\alpha x}$ leaves the group unchanged: the only non-vanishing structure constant is $C_{12}{}^2$ and (9) shows that this does not change. We can try, therefore, to transform I_1 and ϵI_2 with an ϵ dependent unitary matrix so that I_1 remain unchanged, the transformed ϵI_2 converge to I_2

$$S_\epsilon{}^{-1}I_1S_\epsilon = I_1 \qquad \epsilon S_\epsilon{}^{-1}I_2S_\epsilon \rightarrow I_2. \tag{11}$$

This is indeed possible: one has to choose $S_\epsilon = R_{\ln\epsilon}$. This commutes with I_1. It follows from the group relation that $S_\epsilon{}^{-1}T(\beta)S_\epsilon = R_{-\ln\epsilon}T(\beta)$ $R_{\ln\epsilon} = T(\beta/\epsilon)$ and one has, hence, as $h \rightarrow 0$

$$\begin{aligned}\epsilon S_\epsilon{}^{-1}I_2S_\epsilon &= \epsilon S_\epsilon{}^{-1}\lim h^{-1}(T(h) - 1)S_\epsilon \\ &= \lim \epsilon h^{-1}(T(h/\epsilon) - 1) = I_2\end{aligned} \tag{11a}$$

Hence $\epsilon S_\epsilon{}^{-1}I_2S_\epsilon$ not only converges to $J_2 = I_2$ but remains equal to it for all ϵ.

It is more surprising, perhaps, to see that the same device is possible also if one contracts the group with respect to the subgroup of the $T(\beta)$. The contracted group is, in this case, the two parametric abelian group. We demand, in this case, that S_ϵ commute with I_β and hence, by (10a), that it be multiplieation with a function of x. Because of S' unitary nature, we can give it the form exp $(if(x, \epsilon))$. Transformation of the ϵI_α of (10a) with this gives

$$\begin{aligned}\epsilon S_\epsilon{}^{-1}I_\alpha S_\epsilon &= \epsilon e^{-if(x,\epsilon)}({}^1/_2 + x\, d/dx)e^{if(x,\epsilon)} \\ &= \epsilon({}^1/_2 + x\, d/dx) + \epsilon ix\, d/dx\, f(x, \epsilon).\end{aligned} \tag{12}$$

The first part of this converges to 0 as it should since (12) should converge to an operator which commutes with ix. The second part converges to $J_2 = ixf'(x) = ig(x)$ if one sets

$$f(x, \epsilon) = \epsilon^{-1}f(x). \tag{12a}$$

Hence, the transformations of the contracted group corresponding to the parameters α, β is multiplication with

$$e^{i\alpha g(x) + i\beta x} \tag{13}$$

which is again a faithful (reducible) representation of the contracted group.

The operators corresponding to finite group elements could have been obtained directly by transforming $O_{\alpha,\epsilon\beta} = T(\epsilon\beta)R_\alpha$ with $R_{\ln\epsilon}$ in the case of contraction with respect to R_α. Similarly, (13) could have been obtained directly by transforming $O_{\epsilon\alpha,\beta} = T(\beta)R_{\epsilon\alpha}$ with $\exp(\epsilon^{-1}if(x))$.

It is not clear how generally one can obtain a faithful representation of the contracted group as a limit of an ϵ dependent transform of a faithful representation of the original group and the substitution (6b) of its parameters. Certainly, the procedure is not applicable to irreducible representations of compact groups or, more generally, if the infinitesimal operators are bounded.

(*b*) *Representations of the Contracted Group from a Sequence of Representations.*—We shall now give a few examples for the second procedure, i.e., obtaining a representation of the contracted group by choosing a sequence of unitary representations $D^{(1)}, D^{(2)}, \ldots, D^{(l)}, \ldots$ so that each of the operators

$$I^{(l)}_{1\nu},\ \epsilon I^{(l)}_{2\mu} \qquad (\nu = 1, 2, \ldots, r;\ \mu = 1, 2, \ldots, n - r) \tag{15}$$

converge to a finite operator as $\epsilon \to 0$ and $l \to \infty$. Alternately, we can ask that the transformation corresponding to finite group elements

$$D^{(l)}(b_{1\nu}, \epsilon b_{2\mu}) \tag{15a}$$

converge to a unitary representation of the contracted group as $\epsilon \to 0$, $l \to \infty$. The b are the parameters of the contracted group; the operator (15) corresponds, in the representation l, to the group element of the original group the parameters a of which are given by (6b). The formulation making use of the finite group elements (15a) is unambiguous because it deals with the convergence of unitary operators; the first one is usually easier to attack directly.

The convergence of the sequences of (15) and (15a) will depend not only on the values which ϵ assumes and on the corresponding representations $D^{(l)}$, it will also depend in which form that representation is assumed. Hence, method (*a*) can be considered as a special case of the present method in which all $D^{(l)}$ are unitary equivalent.

The contracted group is always an open group because the variability domain of the b is infinite. Hence its representations are, as a rule, infinite dimensional. If the $D^{(l)}$ are finite dimensional, they should be considered to affect only a finite number of the coordinates of Hilbert space.

The simplest non-commutative compact group is the three-dimensional rotation group. All its subgroups are one parametric, we shall contract it with respect to the rotations about the z axis of a rectangular coordinate

system of ordinary space. The contracted group is the Euclidean group of the plane, i.e., the inhomogeneous two-dimensional rotation group. We shall choose for $D^{(l)}$ the representation which is usually denoted[2] by $D^{(l)}$: it is $2l + 1$ dimensional (l being any integer) and is usually described in a space the coordinate axes of which are labeled with $m = -l, -l + 1, \ldots, l - 1, l$. Hence, we label the coordinate axes of Hilbert space with all integers m from $-\infty$ to ∞. In keeping with our previous notation, we call the infinitesimal element which corresponds to rotations about the z axis $M_z = I_1$. One can then write for $|m| \leq l$, $|m'| \leq l$

$$(I_1^{(l)})_{mm'} = (M_z^{(l)})_{mm'} = im\delta_{mm'} \tag{16}$$

$$(I_{2x}^{(l)})_{mm'} = (M_x^{(l)})_{mm'} = -\frac{1}{2}\sqrt{(l-m)(l+m')}\,\delta_{m'm+1} + \frac{1}{2}\sqrt{(l-m')(l+m)}\delta_{m'm-1}$$

$$(I_{2y}^{(l)})_{mm'} = (M_y^{(l)})_{mm'} = \frac{i}{2}\sqrt{(l-m)(l+m')}\,\delta_{m'm+1} + \frac{i}{2}\sqrt{(l-m')(l+m)}\,\delta_{m'm-1}.$$

All matrix elements vanish if either $|m|$ or $|m'|$ is larger than l. As $l \to \infty$, the I_1 converge to a definite operator

$$(J_1)_{mm'} = im\,\delta_{mm'}. \qquad (-\infty < m, m' < \infty) \tag{17a}$$

In fact, the convergence is strong in the sense that $I_1\varphi$ converges strongly to $J_1\varphi$ if φ is in the definition domain of J_1. If ϵ and l converge to zero and infinity, respectively, in such a way that $l\epsilon \to \Xi$, the other two infinitesimal elements will also converge to J_{2x}, J_{2y}, respectively, where

$$(J_{2x})_{mm'} = \frac{1}{2}\Xi(\delta_{m'm-1} - \delta_{m'm+1}) \tag{17b}$$

$$(J_{2y})_{mm'} = \frac{i}{2}\Xi\,(\delta_{m'm-1} + \delta_{m'm+1}).$$

It follows from the fact that there are only a finite number (two) nonvanishing matrix elements in both I_{2x} and I_{2y} that the operators satisfy the commutation relations of the contracted group

$$[J_{2x}, J_{2y}] = 0 \qquad [J_1, J_{2x}] = J_{2y} \qquad [J_1, J_{2y}] = -J_{2x}. \tag{18}$$

It further follows from

$$(I_1^{(l)})^2 + (I_{2x}^{(l)})^2 + (I_{2y}^{(l)})^2 = -l(l+1)$$

by multiplication with ϵ^2 and going to the limit in the above way that

$$J_{2x}^2 + J_{2y}^2 = -\Xi^2. \tag{18a}$$

It was important, for obtaining convergent sequences of $I_1^{(l)}$ and $\epsilon I_2^{(l)}$, to have assumed the $D^{(l)}$ in the form given by (16). This form was reduced out with respect to the subgroup S and this caused, in this case, the convergence of the sequence $I_1^{(l)}$. The convergence of the $\epsilon I_2^{(l)}$ does not actually follow from the convergence of the $I_1^{(l)}$, and hence from the reduced out form of the representations $D^{(l)}$, but was made at least possible by this circumstance.

Before going over to the investigation of the Lorentz groups, it may be worth while to make a final remark about the above contraction, even though it has little to do with our subject. We shall determine, first, the matrices which correspond to finite group elements. For this purpose, it is useful to consider that form of the above representation in which the Hilbert space consists of functions of x and y and the infinitesimal operators have the natural form for infinitesimal operators of the Euclidean group (α and r are polar coordinates $x = r \cos \alpha$, $y = r \sin \alpha$)

$$J_1' = -\partial/\partial\alpha = y\partial/\partial x - x\partial/\partial y \tag{19}$$

$$J_{2x}' = \partial/\partial y \qquad J_{2y}' = -\partial/\partial x. \tag{19a}$$

One should keep in mind that J_{2x}' arose from I_{2x} which is the infinitesimal rotation about the x axis and corresponds to a displacement in the $-y$ direction. Similarly J_{2y}' corresponds to a displacement in the x direction.

The function $\varphi(\alpha, r)$ corresponds in the new Hilbert space to the vector which has the components φ_m in the Hilbert space of (17a), (17b). It further follows from (18a) that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \varphi = -\Xi^2 \varphi \tag{20}$$

whence one can write

$$\begin{aligned} \varphi(x, y) &= \int e^{-i\Xi(x \cos \alpha' + y \sin \alpha')} g(\alpha')d\alpha' \\ &= \int e^{-i\Xi r \cos (\alpha - \alpha')} g(\alpha')d\alpha' \\ &= \int e^{-i\Xi r \cos \alpha'} g(\alpha - \alpha')d\alpha'. \end{aligned} \tag{20a}$$

All integrations are from 0 to 2π. Expanding $g(\alpha - \alpha')$ into a Fourier series of $\alpha - \alpha'$, one finds

$$\varphi(x, y) = \int e^{-i\Xi r \cos \alpha'} \sum g_m e^{-im(\alpha - \alpha')} d\alpha' \tag{20b}$$

m runs from $-\infty$ to ∞. The last form makes it easy to calculate J_1' by (19), the first form permits one to calculate J_{2x}' and J_{2y}' easily. Comparison of the expressions obtained in this way with (17a) and (17b) shows that $g_m = \varphi_m$. In order to remain in keeping with the usual notation, we define[3]

$$2\pi J_m(z) = i^m \int e^{-iz\cos\alpha'} e^{im\alpha'} d\alpha'.$$

J_m is then the ordinary Bessel function of order m. This permits one to write for (20b)

$$\varphi(\alpha, r) = 2\pi \sum \varphi_m i^{-m} e^{-im\alpha} J_m(\Xi r). \tag{21}$$

Because of (19a), one can write down at once the finite group operations. In particular, for the displacement $T(\xi, \eta)$ by ξ and η in the x and y directions one has

$$T(\xi, \eta)\varphi(x, y) = \varphi(x - \xi, y - \eta). \tag{22}$$

Hence, denoting the matrix for the same operation in the original Hilbert space by $T(\xi, \eta)_{mm'}$, one has

$$\varphi(x - \xi, y - \eta) = 2\pi \sum_m \sum_m T(\xi, \eta)_{mm'} \varphi_{m'} i^{-m} e^{-im\alpha} J_m(\Xi r). \tag{22a}$$

This permits an explicit determination of the $T(\xi, \eta)_{mm}'$. We shall not carry this out completely but set only $r = 0$ in (22a). Since all $J_m(0) = 0$ except $J_0(0) = 1$, the summation over m disappears on the right side. The left side becomes, at the same time by (21)

$$\varphi(-\xi, -\eta) = 2\pi \sum \varphi_m i^{-m} e^{-im(\beta + \pi)} J_m(\Xi\rho) \tag{22b}$$

where β, ρ are the polar coordinates for ξ, η. Comparing (22a) and (22b) one finds, with $\xi = 0$, $\beta = {}^1/_2\pi$, $\eta = \rho$

$$J_m(\Xi\rho) = T(0, \rho)_{0m}. \tag{23}$$

The group relations and the form of the infinitesimal operators (19a) gives at once the most important relations for Bessel functions, such as the addition theorem, differential equation (cf. (20)), etc. Up to this point the argument is not new but merely a repetition, for the two dimensional Euclidean group, of a similar reasoning given before[4] for the rotation group. This led to the equation[4]

$$D^{(l)}(0, \beta 0)_{0m} = \bar{P}^m(\cos\beta) = \left(\frac{(l - m)!}{(l + m)!}\right)^{1/2} P_l^m(\cos\beta) \tag{24}$$

in which P_l^m is Legendre's associated function of the first kind, $\bar{P}_l^m$ is normalized to the same value as $P_l^0 = P_l$. Furthermore, 0, β, 0 is the rotation about the x axis so that, by the definition of $T(0, \rho)$

$$T(0, \rho)_{m'm} = (-)^{m'} \lim D^{(l)} \quad (0, \Xi\rho/l, 0)_{m'm}. \tag{25}$$

This, together with (23), gives the asymptotic expression for the associated Legendre functions[5]

$$\lim_{l \to \infty} \bar{P}_l^m(\cos(\rho/l)) = J_m(\rho). \tag{25a}$$

III. Contraction of Lorentz Groups

Let us consider, first, the inhomogeneous Lorentz group with one space-like, one time-like dimension. It is given by the transformations

$$\begin{aligned} x' &= x \cosh \lambda + t \sinh \lambda + a_x \\ t' &= x \sinh \lambda + t \cosh \lambda + a_t. \end{aligned} \tag{26}$$

We wish to contract it with respect to the subgroup of time displacements $t' = t + a_t$. The infinitesimal elements of (26) are: time displacement I_1, space displacement I_{2x} and "rotation" in space-time $I_{2\lambda}$. Their commutation relations read

$$[I_1, I_{2x}] = 0 \qquad [I_1, I_{2\lambda}] = -I_{2x} \qquad [I_{2x}, I_{2\lambda}] = -I_1. \tag{27}$$

Hence, by (9), the commutation relations of the contracted group are

$$[J_1, J_{2x}] = 0 \qquad [J_1, J_{2\lambda}] = -J_{2x} \qquad [J_{2x}, J_{2\lambda}] = 0. \tag{27a}$$

The "rotations" in space-time, together with the displacements in space, form a commutative invariant subgroup.

The matrices

$$\begin{Vmatrix} \cosh \lambda & \sinh \lambda & a_x \\ \sinh \lambda & \cosh \lambda & a_t \\ 0 & 0 & 1 \end{Vmatrix} \tag{26a}$$

form a natural, though not unitary, representation of the group of transformations (26). We can carry out the contraction by setting $a_t = b_t$, $\lambda = \epsilon v$, $a_x = \epsilon b_x$ or $\lambda = v/c$, $a_x = b_x/c$ and letting ϵ converge to 0, or c converge to infinity. If we do this directly in (26a), the representation will not remain faithful for the contracted group. We shall transform therefore (26a) with a suitable ϵ (or c) dependent matrix: multiply the first row with c, the first column with $1/c$. If c goes to infinity in the matrix obtained in this way, one obtains the transformations of the contracted group

$$\begin{aligned} x' &= x + vt + b_x \\ t' &= \qquad t + b_t. \end{aligned} \tag{27a}$$

It is the inhomogeneous Galilei group with one spatial dimension. The transformations $x' = x + vt + b_x$, $t' = t$ form the commutative invariant subgroup.

The same contraction can be carried out for an inhomogeneous Lorentz group with an arbitrary number of spatial dimensions. The only difference is that the subgroup S, with respect to which the contraction is carried out, contains not only the displacements in time as in the above example, but also all purely spatial rotations, i.e., all homogeneous transformations

which leave t invariant. The invariant subgroup of the contracted group consists of all spatial displacements and Galilei transformations: $x_i' = x_i' + v_i t + b_i$, $t' = t$.

Contraction of the Unitary Representations of the Lorentz Groups.—We shall be principally concerned here with the group of the special theory of relativity, i.e., the inhomogeneous Lorentz group with three space-like and one time-like dimension. The subgroup S with respect to which we shall contract it contains the displacements in time, the spatial rotations, and the products of these operations. The contracted group is the ordinary Galilei group, i.e., the group of classical mechanics.

We shall denote the displacement operators in the direction of the three space-like axes by I_k ($k = 1, 2, 3$), the displacement in the direction of the time axis by I_0. The rotations in the kl plane will be denoted by I_{kl}, the acceleration in the direction of the k axis by I_{k0}. I_0 and I_{kl} span the subgroup S.

The quantity

$$I_1^2 + I_2^2 + I_3^2 - I_0^2 = P \tag{28}$$

is a constant in every irreducible representation and the irreducible representations can be divided into three classes according to the value of this constant.[6] In the first class, $P < 0$ and the momenta (which are $-i$ times the infinitesimal operators) are space-like. In the second class $P = 0$ and the momenta form a null vector, $P > 0$ in the third class.

It is generally admitted that the representations of the first class have no physical significance because the momenta of all observed particles are time-like or null vectors. Hence we shall investigate only the simplest one of these representations. Its operators are most easily given in the Hilbert space of functions of three variables p_1, p_2, p_3 which are restricted to the outside of a sphere of radius $\sqrt{-P}$. The expressions for the infinitesimal operators are

$$I_k = ip_k \qquad I_0 = -i(p_1^2 + p_2^2 + p_3^2 + P)^{1/2}. \tag{28a}$$

This last equation also shows the reason for the variables p_k to be restricted by $p_1^2 + p_2^2 + p_3^2 > -P$: if this inequality is not fulfilled, I_0 ceases to be skew hermitean and, hence, the representation is not unitary. Further

$$I_{kl} = p_l\, \partial/\partial p_k - p_k\, \partial/\partial p_l \tag{28b}$$

and

$$I_{k0} = -(p_1^2 + p_2^2 + p_3^2 + P)^{1/2}\, \partial/\partial p_k. \tag{28c}$$

It is useful to introduce new variables instead of the p_1 both in order to simplify the definition domain of the variables and also to bring the operations of the subgroup S into a form which is independent of P. This can

be done, most simply, by introducing as new variables

$$p_0 = (p_1^2 + p_2^2 + p_3^2 + P)^{1/2} \qquad \Omega_k = p_k/(p_0^2 - P)^{1/2}. \tag{29}$$

The Ω_k are restricted to a unit sphere, p_0 changes from 0 to ∞. In these variables, the scalar product between φ and ψ reduces to

$$(\varphi, \psi) = \int d\Omega \int dp_0 (p_0^2 - P)^{1/2} \bar{\varphi}\psi \tag{29a}$$

$d\Omega$ being the surface element of the unit sphere over which the integration with respect to the Ω is to be extended. In terms of the new variables, the infinitesimal operators assume the form

$$I_{kl} = \Omega_l \, \partial/\partial\Omega_k - \Omega_k \, \partial/\partial\Omega_l \qquad I_0 = -ip_0 \tag{30}$$

and

$$I_k = i\Omega_k (p_0^2 - P)^{1/2} \tag{30a}$$

$$I_{k0} = -(p_0^2 - P)^{1/2}\Omega_k \, \partial/\partial p_0 - \frac{p_0}{(p_0^2 - P)^{1/2}} \sum_l \Omega_l (\Omega_l \, \partial/\partial\Omega_k - \Omega_k \, \partial/\partial\Omega_l).$$

If we now set $J_k = \epsilon I_k$, the J_k will converge to zero unless $-P$ becomes inversely proportional to ϵ^2, i.e., unless $-\epsilon^2 P$ converges to a definite limit P. If this is assumed, the second term of ϵI_{k0} will converge to zero and the infinitesimal elements of the representation of the contracted group become

$$J_{kl} = \Omega_l \, \partial/\partial\Omega_k - \Omega_k \, \partial/\partial\Omega_l \qquad J_0 = -ip_0 \tag{31}$$

$$J_{k0} = -\mathrm{P}\Omega_k \, \partial/\partial p_0 \qquad J_k = i \, \mathrm{P}\Omega_k. \tag{31a}$$

These operators indeed span a unitary representation of the Galilei group: they correspond to case II with $m = 0$ of a recent determination of these representations.[7] The p_k of this article correspond to our $\mathrm{P}\Omega_k$, the variable s is given by $i \, \mathrm{P} \, \partial/\partial p_0$. One also understands now why it was impossible to find a physical interpretation to this representation: it is the limiting case of a representation of the relativistic group with imaginary mass $P^{1/2}$. The same is probably true of the other true representations[7] of the Galilei group.

We go over now to the investigation of the simplest representation with positive P, i.e., the representation of the Klein-Gordon equation. The infinitesimal operators are again given by (28a), (28b), (28c). However, since P is positive, the variability domain of the p extends over the whole three dimensional space. The scalar product of two functions φ and ψ is now given by

$$(\varphi, \psi) = \int\int\int dp_1 \, dp_2 \, dp_3 \, p_0^{-1} \, \bar{\varphi}\psi \tag{32}$$

in which p_0 is still given by (29).

If we set, in the sense of (6),

$$J_k = \epsilon I_k = \epsilon i p_k \tag{33}$$

the operation of time displacement becomes

$$iJ_0 = iI_0 = (-I_1^2 - I_2^2 - I_3^2 + P)^{1/2} = \epsilon^{-1}(-J_1^2 - J_2^2 - J_3^2 + \epsilon^2 P)^{1/2} \tag{33a}$$

Since the J_k are skew hermitean, their squares are positive definite hermitean operators. Since $\epsilon^2 P$ is also positive the expectation value $(\varphi, iJ_0\varphi)$ of iJ_0 is, for any state φ, greater than ϵ^{-1} times the expectation value of iJ_1. It follows that if $J_1\varphi$ converges to a vector in Hilbert space as $\epsilon \to 0$, the vector $J_0\varphi$ must grow beyond all limits. The same is true, of course, for the other J_k. It follows that the representations considered cannot be contracted in the sense discussed in the previous sections and the same is true of all representations of the classes $P \geq 0$.

It is possible, however, to contract these representations to representations up to a factor of the Galilei group. The commutation relations of the infinitesimal elements of representations up to a factor differ from the commutation relations of real representations by the appearance of a constant in the structural relations. Hence

$$[J_\alpha, J_\beta] = \sum c_{\alpha\beta}^\gamma J_\gamma + b_{\alpha\beta}1 \tag{34}$$

where $c_{\alpha\beta}^\gamma$ are the structure constants of the group to be represented (in our case, the inhomogeneous Galilei group) and the $b_{\alpha\beta}1$ are multiples of the unit operator. One will, therefore, obtain infinitesimal elements of representations up to a factor if one sets, instead of (6)

$$J_{1\nu} = I_{1\nu} - a_{1\nu}1 \qquad J_{2\nu} = \epsilon I_{2\nu} - a_{2\nu}1 \tag{34a}$$

in which all a may depend on ϵ. Since the additional terms in (34a) commute with all other operators, these additional terms will not affect the left side of (34). Hence, they must be compensated also on the right side and this is done by the additional terms $b_{\alpha\beta}1$. The point of introducing the terms $a1$ in (34a), which then necessitates the introduction of the b in (34), is that the right sides of (34a) may converge to finite non-vanishing operators even if the $I_{1\nu}$, $\epsilon I_{2\nu}$ cannot be made to converge.

The above generalization of the concept of contraction indeed allows a contraction of the representations given by (28a), (28b), (28c) also for $P > 0$. As we let P go to infinity, I_0 will also tend to infinity ($(I_0\varphi, I_0\varphi)$ converges to infinity for all φ). However, subtracting $-iP^{1/2}1$ from I_0, it will converge to

$$\begin{aligned} J_0 &= \lim -i(-I_1^2 - I_2^2 - I_3^2 + P)^{1/2} + iP^{1/2} \\ &= (i/2P^{1/2})(I_1^2 + I_2^2 + I_3^2) = (i/2P^{1/2}\epsilon^2)(J_1^2 + J_2^2 + J_3^2). \end{aligned} \tag{35}$$

This shows that J_0 will converge to a finite operator if the $J_k = \epsilon I_k$ do and if $P^{1/2}\epsilon^2$ converges to a finite constant m as $P \to \infty$, $\epsilon \to 0$. Both can be accomplished by assuming the representation in such a form that, instead of (28)

$$I_k = ip_k/\epsilon \quad I_0 = -i(P + (p_1^2 + p_2^2 + p_3^2)/\epsilon^2)^{1/2}. \tag{36}$$

This is indeed possible because the variability domain of the p_k is unrestricted and the above form of the infinitesimal elements can be obtained by unitary transformation of the operators given in (28a). Such a transformation leaves the I_{kl} of (28b) unchanged but transforms the I_{k0} of (28c) into

$$I_{k0} = -\epsilon(P + (p_1^2 + p_2^2 + p_3^2)/\epsilon^2)^{1/2}\partial/\partial p_k. \tag{36b}$$

Hence we shall have

$$\begin{aligned} J_0 &= \lim -i(P + (p_1^2 + p_2^2 + p_3^2)/\epsilon^2)^{1/2} + iP^{1/2} = \\ &\qquad -(i/2m)(p_1^2 + p_2^2 + p_3^2) \\ J_{kl} &= p_l\partial/\partial p_k - p_k\partial/\partial p_l \end{aligned} \tag{37.1}$$

$$\begin{aligned} J_k &= \lim \epsilon(ip_k/\epsilon) = ip_k \\ J_{k0} &= \lim -\epsilon(P + (p_1^2 + p_2^2 + p_3^2)/\epsilon^2)^{1/2}\epsilon\partial/\partial p_k \\ &= \lim -(P\epsilon^4 + \epsilon^2(p_1^2 + p_2^2 + p_3^2))^{1/2}\partial/\partial p_k = -m\partial/\partial p_k. \end{aligned} \tag{37.2}$$

The reader familiar with the transition from the Klein-Gordon to the Schrödinger equation will recognize the increase of the rest mass with increasing c and the elimination of this rest mass by the subtraction of $-iP^{1/2}1$ from the infinitesimal operator of the time-displacement operator. The infinitesimal operators (37.1), (37.2) for the contracted group are in fact those of Schrödinger's theory. It is likely that a similar contraction is possible also for the other representations with positive rest mass (i.e., $P > 0$) but this and the behavior of the representations with $P = 0$ will not be further discussed here.

[1] Segal, I. E., *Duke Math. J.*, **18**, 221 (1951).

[2] Cf. e.g. Wigner, E., *Gruppentheorie und ihre Auwendiengen* etc., Friedr. Vieweg, Braunschweig (1931) and Edwards Brothers, Ann Arbor (1944). Chapter XV.

[3] Cf. Jahnke, E., and Emde, F., *Tables of Functions*, Dover Publications, 1943, p. 149; or Watson, G. N., *Treatise on Bessel Functions*, Cambridge Univ. Press, 1922, p. 19ff.

[4] Reference 2, Chapter XIX, particularly p. 230, 232. Cf. also Wigner, E. P., *J. Franklin Inst.*, **250**, 477 (1950), and Godement, R., *Trans. Amer. Math. Soc.* **73**, 496 (1952).

[5] For $m = 0$ this is given on p. 65, of Watson's *Bessel Functions* (ref. 3).

[6] Cf. e.g. Bargmann, V., and Wigner, E. P., these PROCEEDINGS, **34**, 211 (1948) and further literature quoted there.

[7] Inonu, E., and Wigner, E. P., *Nuovo cimento*, **9**, 705 (1952).