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Mathematics

Calculus of twisted vertex operators

(affine Lie algebras/basic modules/string theory)

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ABSTRACT **Starting from an arbitrary isometry of an arbitrary even lattice, twisted and shifted vertex operators are introduced. Under commutators, these operators provide realizations of twisted affine Lie algebras. This construction, generalizing a number of known ones, is based on a self-contained "calculus."**

Section 1. Introduction

The discovery that vertex operators can provide constructions of affine Lie algebras has stimulated much activity. In the first construction (1) and its generalization (2), affine algebras of types A, D, and E, twisted by the principal automorphism of the underlying finite-dimensional simple Lie algebra $\mathfrak{g}$, were realized by means of certain "twisted vertex operators." (The principal automorphism extends the Coxeter element of the Weyl group, acting on a Cartan subalgebra.) In the next construction (refs. 3 and 4), untwisted affine algebras of types $A^{(1)}$, $D^{(1)}$, and $E^{(1)}$ were represented using the "untwisted vertex operators" of dual-string theory. This was modified in ref. 5 to obtain twistings by certain outer involutions of $\mathfrak{g}$. In ref. 6, another twisted construction was found, this time based on an involution of $\mathfrak{g}$ that is -1 on a Cartan subalgebra, providing a generalization of ref. 1 different from that of ref. 2.

Certain aspects of these constructions were uniformly generalized in ref. 7, in which the role of an automorphism of the Cartan subalgebra was emphasized. Here I continue the approach of ref. 7 to give a common generalization of all the existing constructions, starting from an arbitrary isometry of an arbitrary even lattice and based entirely on elementary calculations with vertex operators. I construct an affine algebra as an explicit algebra of operators on a "generalized Fock space" and, in the process, obtain a formula for an extension to $\mathfrak{g}$ of the automorphism of the root system. The case of the principal automorphism is analyzed in detail in ref. 8.

I also give a "shifted" generalization of the construction. Over $\mathbb{C}$, I thus obtain continuous families of twisted realizations of affine Lie algebras of types A, D, and E. When the "shifting parameter" is rational, the resulting twisted affine algebra has a corresponding $\mathbb{Z}$-grading.

One of the motivations was to try to find an algebraic approach to reducing the critical space-time dimension in string theory from 26 or 10 to 4. In our construction, the dimension of the span of the lattice is "reduced" to the dimension of the span of the sublattice fixed by the automorphism. When applied to the E_8 root lattice, perhaps this can be used to break the E_8-symmetry of the heterotic string (9).

The details of this work will appear elsewhere.

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Section 2. Assumptions

Suppose the following.

2.1. L is a finitely generated free abelian group.

2.2. $\langle\cdot,\cdot\rangle$ is a nonsingular symmetric $\mathbb{Z}$-bilinear form on L such that

$$\langle\alpha,\alpha\rangle \in 2\mathbb{Z} \quad \text{for} \quad \alpha \in L.$$

2.3. ν is an automorphism of L such that $\langle\nu\alpha,\nu\beta\rangle = \langle\alpha,\beta\rangle$ for $\alpha, \beta \in L$.

2.4. m is a positive integer such that $\nu^m = 1$.

2.5. If m is even, $\langle\nu^{m/2}\alpha,\alpha\rangle \in 2\mathbb{Z}$ for $\alpha \in L$.

Remark: Given L, $\langle\cdot,\cdot\rangle$, ν and m satisfying assumptions 2.1–2.4, assumption 2.5 can always be arranged by doubling m if necessary.

Observe that

$$\left\langle \sum_{p\in\mathbb{Z}/m\mathbb{Z}} \nu^p\alpha, \alpha \right\rangle \in 2\mathbb{Z} \text{ for } \alpha \in L. \qquad [2.1]$$

Section 3. Notation

Let $\mathbb{F}$ be a field of characteristic 0 containing a primitive mth root of unity ω. (Later we shall assume that $\mathbb{F}$ contains additional roots of unity. We may take $\mathbb{F} = \mathbb{C}$, $\omega = e^{2\pi i/m}$.) Embed L canonically in the $\mathbb{F}$-vector space $\mathfrak{h}$, where $\mathfrak{h} = \mathbb{F}\otimes_{\mathbb{Z}} L$, and extend $\langle\cdot,\cdot\rangle$ and ν by $\mathbb{F}$-linearity to $\mathfrak{h}$.

We shall use some elementary results in the self-contained sections 2 and 3 of ref. 7.

For $n \in \mathbb{Z}$, set $\mathfrak{h}_{(n)} = \{x \in \mathfrak{h} | \nu x = \omega^n x\} \subset \mathfrak{h}$, so that $\mathfrak{h} = \amalg_{p\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{h}_{(p)}$. (We identify $\mathfrak{h}_{(n \bmod m)}$ with $\mathfrak{h}_{(n)}$ for $n \in \mathbb{Z}$.) For $p \in \mathbb{Z}/m\mathbb{Z}$, denote by

$$P_p : \mathfrak{h} \to \mathfrak{h}_{(p)} \qquad [3.1]$$

the pth projection and, for $x \in \mathfrak{h}$ and $n \in \mathbb{Z}$, set $x_{(n)} = P_{(n \bmod m)}x$. Viewing $\mathfrak{h}$ as an abelian Lie algebra, consider the ν-twisted affine Lie algebra (in the terminology of ref. 10) $\tilde{\mathfrak{h}}(\nu) = \amalg_{n\in\mathbb{Z}} \mathfrak{h}_{(n)} \otimes t^n \oplus \mathbb{F}c \oplus \mathbb{F}d$ with brackets determined by

$$[x\otimes t^i, y\otimes t^j] = m^{-1}\langle x, y\rangle i\delta_{i+j,0}c \qquad [3.2]$$
$$[c, \tilde{\mathfrak{h}}(\nu)] = 0,\ [d, x\otimes t^i] = ix\otimes t^i$$

for $i, j \in \mathbb{Z}$, $x \in \mathfrak{h}_{(i)}$, $y \in \mathfrak{h}_{(j)}$. Consider also its commutator subalgebra $\tilde{\mathfrak{h}}(\nu)' = \amalg_{n\neq 0} \mathfrak{h}_{(n)} \otimes t^n \oplus \mathbb{F}c$ and the subalgebras $\hat{\mathfrak{h}}(\nu)_\pm = \amalg_{\pm n>0} \mathfrak{h}_{(n)} \otimes t^n$, $\mathfrak{b}(\nu) = \hat{\mathfrak{h}}(\nu)_+ \oplus \mathfrak{h}_{(0)} \oplus \mathbb{F}c \oplus \mathbb{F}d$. The form $\langle\cdot,\cdot\rangle$ being nonsingular on $\mathfrak{h}$, we observe that $\tilde{\mathfrak{h}}(\nu)'$ is a Heisenberg Lie algebra, in the sense that its commutator subalgebra equals its center and is one-dimensional.

Make $\mathbb{F}$ a (one-dimensional) $\mathfrak{b}$-module as follows: $\hat{\mathfrak{h}}(\nu)_+\cdot 1 = 0$, $\mathfrak{h}_{(0)}\cdot 1 = 0$, $c\cdot 1 = 1$, $d\cdot 1 = 0$. Form the following induced

$\tilde{\mathfrak{h}}(\nu)$-module, which we denote S:

$$S = S(\hat{\mathfrak{h}}(\nu)_-) = U(\tilde{\mathfrak{h}}(\nu)) \otimes_{U(\mathfrak{b})} \mathbb{F}. \qquad [3.3]$$

Here $U(\cdot)$ denotes the universal enveloping algebra, $S(\cdot)$ denotes the symmetric algebra, and we have used the Poincaré–Birkhoff–Witt theorem to make the identification. Then S is irreducible even under the Heisenberg subalgebra $\tilde{\mathfrak{h}}(\nu)'$, and S provides the "canonical realization of the Heisenberg commutation relations." The action of d defines a $\mathbb{Z}$-grading on S: $S = \Pi_{n\in\mathbb{N}} S_n$, where S_n is the n-eigenspace of d and $\mathbb{N}$ denotes $\{0, 1, 2, \ldots\}$.

For $n \in \mathbb{Z}$ and $x \in \mathfrak{h}_{(n)}$, write $x(n)$ for the operator on S corresponding to $x \otimes t^n$.

We shall use the elementary calculus of formal variables ζ, $\zeta_1, \zeta_2, \ldots$ explained in sections 2 and 3 of ref. 7. For $\alpha \in \mathfrak{h}$, define $E^{\pm}(\alpha, \zeta) = \exp(\Sigma_{\pm n>0}\, m\alpha_{(n)}(n)\zeta^n/n)$, a pair of formal Laurent series in ζ with coefficients in End S. (Here exp denotes the formal exponential series.) By proposition 3.4 of ref. 7, we have

$$E^+(\alpha, \zeta_1)E^-(\beta, \zeta_2) = E^-(\beta, \zeta_2)E^+(\alpha, \zeta_1) \prod_{p\in\mathbb{Z}/m\mathbb{Z}} (1 - \omega^{-p}\zeta_1/\zeta_2)^{\langle \nu^p\alpha, \beta\rangle} \qquad [3.4]$$

for $\alpha, \beta \in \mathfrak{h}$.

Section 4. An Identity

Define the function

$$\begin{aligned} C: L \times L &\to \mathbb{F}^\times \\ (\alpha, \beta) &\mapsto (-1)^{\langle \Sigma\nu^p\alpha, \beta\rangle}\omega^{\langle \Sigma p\nu^p\alpha, \beta\rangle} \\ &= \Pi(-\omega^p)^{\langle \nu^p\alpha, \beta\rangle}, \end{aligned} \qquad [4.1]$$

where the sums and product range over $p \in \mathbb{Z}/m\mathbb{Z}$. Then C is bilinear into the abelian group $\mathbb{F}^\times$; i.e.,

$$\begin{aligned} C(\alpha + \beta, \gamma) &= C(\alpha, \gamma)C(\beta, \gamma) \\ C(\alpha, \beta + \gamma) &= C(\alpha, \beta)C(\alpha, \gamma) \end{aligned} \qquad [4.2]$$

for $\alpha, \beta, \gamma \in L$. Also,

$$C(\alpha, \alpha) = 1 \qquad [4.3]$$

$$C(\nu\alpha, \nu\beta) = C(\alpha, \beta). \qquad [4.4]$$

The verification of [4.3] uses [2.1]. Note that $C(\beta, \alpha) = C(\alpha, \beta)^{-1}$.

Recall from section 2 of ref. 7 the notation $\delta(\zeta) = \Sigma_{n\in\mathbb{Z}}\zeta^n$, $D\delta(\zeta) = \Sigma_{n\in\mathbb{Z}}n\zeta^n$, where $D = D_\zeta = \zeta(d/d\zeta)$. For $\alpha, \beta \in L$, set $\varepsilon_1(\alpha, \beta) = \Pi_{0<p<m}(1 - \omega^{-p})^{\langle \nu^p\alpha, \beta\rangle}$,

$$F(\alpha, \beta) = \frac{1}{2}\sum_{0<p<m} \frac{\langle(\nu^p - \nu^{-p})\alpha, \beta\rangle}{1 - \omega^p}.$$

Note that ε_1 (resp, F) is a bilinear function from $L \times L$ to $\mathbb{F}^\times$ (resp, $\mathbb{F}$). Also,

$$\begin{aligned} \varepsilon_1(\nu\alpha, \nu\beta) &= \varepsilon_1(\alpha, \beta), \\ \varepsilon_1(\alpha, \beta)/\varepsilon_1(\beta, \alpha) &= (-1)^{\langle\alpha,\beta\rangle}C(\alpha, \beta)^{-1}, \\ F(\alpha, \alpha) &= 0. \end{aligned} \qquad [4.5]$$

PROPOSITION 4.1. *Let $\alpha, \beta \in L$. Set*

$$\mathrm{I}(\mathrm{n}) = \{\mathrm{p} \in \mathbb{Z}/\mathrm{m}\mathbb{Z} | \langle \nu^{\mathrm{p}}\alpha, \beta\rangle = \mathrm{n}\}$$

for $\mathrm{n} \in \mathbb{Z}$, and suppose that $\langle \nu^{\mathrm{p}}\alpha, \beta\rangle \geq -2$ for all $\mathrm{p} \in \mathbb{Z}/\mathrm{m}\mathbb{Z}$. Then

$$\begin{aligned} &\Pi(1 - \omega^{-\mathrm{p}}\zeta_1/\zeta_2)^{\langle \nu^{\mathrm{p}}\alpha, \beta\rangle} \\ &\qquad - \mathrm{C}(\beta, \alpha)(\zeta_1/\zeta_2)^{\langle \Sigma\nu^{\mathrm{p}}\alpha, \beta\rangle}\Pi(1 - \omega^{-\mathrm{p}}\zeta_2/\zeta_1)^{\langle \nu^{\mathrm{p}}\beta, \alpha\rangle} \\ &= \sum_{\mathrm{r}\in\mathrm{I}(-1)} \varepsilon_1(\nu^{\mathrm{r}}\alpha, \beta)\delta(\omega^{-\mathrm{r}}\zeta_1/\zeta_2) + \sum_{\mathrm{r}\in\mathrm{I}(-2)} \varepsilon_1(\nu^{\mathrm{r}}\alpha, \beta)\Big[\mathrm{D}\delta(\omega^{-\mathrm{r}}\zeta_1/\zeta_2) \\ &\qquad - \left(\frac{1}{2}\left\langle \sum \nu^{\mathrm{p}}\alpha, \beta\right\rangle + \mathrm{F}(\nu^{\mathrm{r}}\alpha, \beta)\right)\delta(\omega^{-\mathrm{r}}\zeta_1/\zeta_2)\Big], \end{aligned}$$

where the unindexed sums and products range over $\mathrm{p} \in \mathbb{Z}/\mathrm{m}\mathbb{Z}$.

We now obtain another description of the function ε_1. For $\alpha, \beta \in L$, define $\varepsilon_2(\alpha, \beta) = (-1)^{\langle \Sigma\nu^p\alpha, \beta\rangle}\omega^{-\langle \Sigma p\nu^p\alpha, \beta\rangle} = \Pi(-\omega^{-p})^{\langle \nu^p\alpha, \beta\rangle}$, where the sums and product range over $-m/2 < p < 0$. Then ε_2 is bilinear, $\varepsilon_2(\nu\alpha, \nu\beta) = \varepsilon_2(\alpha, \beta)$ and

$$\varepsilon_2(\alpha, \beta)/\varepsilon_2(\beta, \alpha) = (-1)^{\langle\alpha,\beta\rangle}C(\alpha, \beta)^{-1}. \qquad [4.6]$$

Write $\varepsilon'(\alpha, \beta) = \Pi_{0<p<m/2}(1 - \omega^{-p})^{\langle(\nu^p + \nu^{-p})\alpha, \beta\rangle}$

$$\varepsilon^*(\alpha, \beta) = \begin{cases} \varepsilon'(\alpha, \beta)2^{\langle \nu^{m/2}\alpha, \beta\rangle} & \text{if } m \in 2\mathbb{Z} \\ \varepsilon'(\alpha, \beta) & \text{if } m \in 2\mathbb{Z} + 1. \end{cases}$$

Then

$$\varepsilon_1(\alpha, \beta) = \varepsilon_2(\alpha, \beta)\varepsilon^*(\alpha, \beta). \qquad [4.7]$$

Recalling assumption 2.5, we define $\sigma'(\alpha) = \Pi_{0<p<m/2}(1 - \omega^{-p})^{\langle \nu^p\alpha, \alpha\rangle}$

$$\sigma(\alpha) = \begin{cases} \sigma'(\alpha)2^{\langle \nu^{m/2}\alpha, \alpha\rangle/2} & \text{if } m \in 2\mathbb{Z} \\ \sigma'(\alpha) & \text{if } m \in 2\mathbb{Z} + 1. \end{cases} \qquad [4.8]$$

Then

$$\sigma(\nu\alpha) = \sigma(\alpha) \qquad [4.9]$$

$$\varepsilon^*(\alpha, \beta) = \sigma(\alpha + \beta)/\sigma(\alpha)\sigma(\beta). \qquad [4.10]$$

Section 5. $\hat{L}$ and $\hat{\nu}$

Set $\omega_0 = (-1)^m\omega$. Then ω_0 is a primitive $2m$th root of unity if m is odd and, for any m, -1 and ω are powers of ω_0. In view of formulas 4.2 and 4.3, there is a unique (up to equivalence) central extension

$$1 \to \langle\omega_0\rangle \to \hat{L} \xrightarrow{-} L \to 1 \qquad [5.1]$$

of L by the cyclic group generated by ω_0 with commutator map C—i.e., such that

$$aba^{-1}b^{-1} = C(\bar{a}, \bar{b}) \quad \text{for} \quad a, b \in \hat{L}. \qquad [5.2]$$

If we replace the map $^-$ in [5.1] by $\nu \circ {}^-$, we obtain another central extension of L by $\langle\omega_0\rangle$ with commutator map C, by [4.4]. By the uniqueness, there is an automorphism $\hat{\nu}$ of $\hat{L}$ (fixing ω_0) such that $\hat{\nu}$ covers ν—i.e., such that

$$(\hat{\nu}a)^- = \nu\bar{a} \quad \text{for} \quad a \in \hat{L}. \qquad [5.3]$$

It is easy to see that the automorphisms of $\hat{L}$ covering the identity automorphism of L are precisely the maps ρ^*: $a \mapsto a\rho(\bar{a})$ for a homomorphism ρ: $L \to \langle\omega_0\rangle$. Similarly, there is a

homomorphism ρ_0: $L \cap \mathfrak{h}_{(0)} \to \langle\omega_0\rangle$ such that $\hat{\nu}a = a\rho_0(\bar{a})$ if $\nu\bar{a} = \bar{a}$. Now ρ_0 can be extended to a homomorphism $\rho: L \to \langle\omega_0\rangle$ since the map $1 - P_0$ induces an isomorphism from $L/L \cap \mathfrak{h}_0$ to the free abelian group $(1 - P_0)L$ (the projection of L to the orthogonal complement of $\mathfrak{h}_{(0)}$). Multiplying $\hat{\nu}$ by the inverse of ρ^* gives us an automorphism $\hat{\nu}$ of $\hat{L}$ satisfying [5.3] and

$$\hat{\nu}a = a \quad \text{if} \quad \nu\bar{a} = \bar{a}. \qquad [5.4]$$

Section 6. Irreducible $\hat{N}$-Modules

Let $N = (1 - P_0)\mathfrak{h} \cap L = \{\alpha \in L | \langle\alpha, \mathfrak{h}_{(0)}\rangle = 0\}$, $M = (1 - \nu)L \subset N$. On N, the commutator map C (see [4.1]) simplifies to

$$C_N(\alpha, \beta) = \omega^{\langle\Sigma p \nu^p \alpha, \beta\rangle}. \qquad [6.1]$$

Let $R = \{\alpha \in N | C_N(\alpha, N) = 1\}$, the radical of the alternating bilinear form C_N. It is easy to see that $M \subset R$. Denoting by $\hat{Q}$ the subgroup of $\hat{L}$ obtained by pulling back a subgroup Q of L (see [5.1]), we observe that $\hat{R}$ is the center of $\hat{N}$ and that $\hat{M} \subset \hat{R}$. Observe that $a\hat{\nu}a^{-1} \in \hat{M}$ for all $a \in \hat{L}$. Using [5.4], we can prove the following.

PROPOSITION 6.1. *There is a unique homomorphism* τ: $\hat{\mathrm{M}} \to \mathbb{F}^\times$ *such that* $\tau(\omega_0) = \omega_0$ *and* $\tau(\mathrm{a}\hat{\nu}\mathrm{a}^{-1}) = \omega^{-\langle\Sigma\nu^p\bar{\mathrm{a}}, \bar{\mathrm{a}}\rangle/2}$ *for* $\mathrm{a} \in \hat{\mathrm{L}}$.

Now N/M is a finite group, and hence so are $\hat{N}/\hat{M}$ and $\hat{N}/\mathrm{Ker}\ \tau$. Choose $m_0 \in \mathbb{N}$ such that $x^{m_0} = 1$ for all $x \in \hat{N}/\mathrm{Ker}\ \tau$.

PROPOSITION 6.2. *Assume that* $\mathbb{F}$ *contains a primitive* m_0*th root of unity. There are exactly* $|\mathrm{R}/\mathrm{M}|$ *extensions of* τ *to a homomorphism* $\chi: \hat{\mathrm{R}} \to \mathbb{F}^\times$. *For each such* χ, *there is a unique (up to equivalence) irreducible* $\hat{\mathrm{N}}$*-module on which* $\hat{\mathrm{R}}$ *acts according to* χ, *and every irreducible* $\hat{\mathrm{N}}$*-module on which* $\hat{\mathrm{M}}$ *acts according to* τ *is equivalent to one of these. Every such module has dimension* $|\mathrm{N}/\mathrm{R}|^{1/2}$. *To construct the module corresponding to* χ, *let* A *be any subgroup of* N *(necessarily containing* R*) that is maximal such that the alternating bilinear form* **6.1** *is trivial on* A. *Then* $\hat{\mathrm{A}}$ *is a maximal abelian subgroup of* $\hat{\mathrm{N}}$. *Let* $\psi: \mathrm{A} \to \mathbb{F}^\times$ *be any homomorphism extending* χ *and denote by* $\mathbb{F}_\psi$ *the* $\hat{\mathrm{A}}$*-module* $\mathbb{F}$ *with character* ψ. *Then* T *is isomorphic to the induced* $\hat{\mathrm{N}}$*-module* $\mathbb{F}[\hat{\mathrm{N}}] \otimes_{\mathbb{F}[\hat{\mathrm{A}}]} \mathbb{F}_\psi \simeq \mathbb{F}[\mathrm{N}/\mathrm{A}]$ ($\mathbb{F}[\cdot]$ *denoting group algebra).*

Section 7. The Vertex Operators

Let T be any $\hat{N}$-module on which $\hat{M}$ acts as multiplication by the character τ (see *Proposition 6.1*). Form the induced $\hat{L}$-module $U_T = \mathbb{F}[\hat{L}] \otimes_{\mathbb{F}[\hat{N}]} T$. Since T may be viewed as a module for the finite group $\hat{N}/\mathrm{Ker}\ \tau$, T is completely reducible. In case $\mathbb{F}$ contains enough roots of unity, the structure of T follows from *Proposition 6.2* and, in the irreducible case, $U_T = \mathbb{F}[\hat{L}] \otimes_{\mathbb{F}[\hat{A}]} \mathbb{F}_\psi \simeq \mathbb{F}[L/A]$.

In general, $\hat{L}$ and $\mathfrak{h}_{(0)}$ act on U_T as follows:

$$a \cdot b \otimes t = ab \otimes t,$$
$$\alpha \cdot b \otimes t = \langle\alpha, \bar{b}\rangle b \otimes t \qquad [7.1]$$

for $a, b \in \hat{L}$, $t \in T$, $\alpha \in \mathfrak{h}_{(0)}$, and we have $[\alpha, a] = \langle\alpha, \bar{a}\rangle a$. In case $\langle\alpha, L\rangle \subset \mathbb{Z}$, define the End U_T-valued formal Laurent series ζ^α as follows:

$$\zeta^\alpha \cdot b \otimes t = \zeta^{\langle\alpha, \bar{b}\rangle} b \otimes t \qquad [7.2]$$

and the operator ω^α on U_T by $\omega^\alpha \cdot b \otimes t = \omega^{\langle\alpha, \bar{b}\rangle} b \otimes t$. Then $\zeta^\alpha a = a\zeta^{\alpha + \langle\alpha, \bar{a}\rangle}$, $\omega^\alpha a = a\omega^{\alpha + \langle\alpha, \bar{a}\rangle}$. Moreover,

$$\hat{\nu}a = a\omega^{-\Sigma\nu^p\bar{a} - \langle\Sigma\nu^p\bar{a}, \bar{a}\rangle/2} \qquad [7.3]$$

on U_T. It follows that $\hat{\nu}^m = 1$ on U_T and hence on $\hat{L}$.

Define a $\mathbb{Z}$-grading on U_T by

$$\deg(b \otimes t) = -\frac{1}{2}\left\langle \sum \nu^p \bar{b}, \bar{b} \right\rangle \qquad [7.4]$$

and define an operator d on $U_T = \amalg_{n \in \mathbb{N}} (U_T)_n$ accordingly. Then U_T becomes an $\tilde{\mathfrak{h}}(\nu)$-module by making $\tilde{\mathfrak{h}}(\nu)'$ act trivially.

Since the projection operator P_0 (see [3.1]) induces an isomorphism from L/N to P_0L, we have a natural isomorphism

$$U_T \simeq \mathbb{F}[P_0L] \otimes_{\mathbb{F}} T \qquad [7.5]$$

of $\tilde{\mathfrak{h}}(\nu)$-modules, $\tilde{\mathfrak{h}}(\nu)$ acting in the obvious way on $\mathbb{F}[P_0L] \otimes T$.

Recalling [3.3], set

$$V_T = S \otimes_{\mathbb{F}} U_T = \coprod_{n \in \mathbb{N}} (V_T)_n, \qquad [7.6]$$

a tensor product $\tilde{\mathfrak{h}}(\nu)$-module on which $\hat{L}$ acts by its action on the second factor and whose $\mathbb{Z}$-grading is determined by the action of d. For $a \in \hat{L}$, we define the corresponding vertex operator $X(a, \zeta)$ and its coefficients $x_a(n) \in \mathrm{End}\ V_T$ as follows:

$$X(a, \zeta) = E^-(-\bar{a}, \zeta)E^+(-\bar{a}, \zeta)a\zeta^{-\Sigma\nu^p\bar{a} - \langle\Sigma\nu^p\bar{a}, \bar{a}\rangle/2} \qquad [7.7]$$
$$= \sum_{n \in \mathbb{Z}} x_a(n)\zeta^n,$$

an End V_T-valued formal Laurent series.

Using [7.3] we obtain

$$X(\hat{\nu}a, \zeta) = X(a, \omega\zeta) \qquad [7.8]$$
$$DX(a, \zeta) = [d, X(a, \zeta)], \qquad [7.9]$$

the last formula showing that each operator $x_a(n)$ has degree n [i.e., takes $(V_T)_m$ to $(V_T)_{m+n}$ for all $m \in \mathbb{Z}$].

It will be convenient to renormalize the vertex operators as follows: Recalling [4.8], define $Y(a, \zeta) = m^{-\langle\bar{a}, \bar{a}\rangle/2}\sigma(\bar{a})X(a, \zeta) = \Sigma_{n \in \mathbb{Z}} y_a(n)\zeta^n$. Then formulas **7.8** and **7.9** also hold with Y in place of X (recall [4.9]).

Section 8. Commutators

For $\alpha \in \mathfrak{h}$, write $\alpha(\zeta) = \Sigma_{n \in \mathbb{Z}} \alpha_{(n)}(n)\zeta^n$, $\alpha^\pm(\zeta) = \Sigma_{n \in \mathbb{Z}, \pm n > 0} \alpha_{(n)}(n)\zeta^n + (1/2)\alpha_{(0)}(0)$, where $\alpha_{(n)}(n)$ is understood as an operator on V_T in the obvious way. Then, $\alpha(\zeta) = \alpha^+(\zeta) + \alpha^-(\zeta)$. For $a \in \hat{L}$ and $x \in \mathbb{F}$, set $:(\alpha(\zeta) + x)Y(a, \zeta): = \alpha^-(\zeta)Y(a, \zeta) + Y(a, \zeta)\alpha^+(\zeta) + xY(a, \zeta)$.

Using [3.4], *Proposition 4.1*, [4.7], [4.10], [5.2], and the rules for multiplying suitable expressions by δ and $D\delta$ (see, e.g., proposition 3.9 of ref. 7), we can prove the following:

THEOREM 8.1. *Let* $\mathrm{a}, \mathrm{b} \in \hat{\mathrm{L}}$ *and suppose that* $\langle\nu^{\mathrm{p}}\bar{\mathrm{a}}, \bar{\mathrm{b}}\rangle \geq -2$ *for all* $\mathrm{p} \in \mathbb{Z}/\mathrm{m}\mathbb{Z}$. *For* $\mathrm{n} \in \mathbb{Z}$, *set* $\mathrm{J}(\mathrm{n}) = \{\mathrm{p} \in \mathbb{Z}/\mathrm{m}\mathbb{Z} | \langle\nu^{\mathrm{p}}\bar{\mathrm{a}}, \bar{\mathrm{b}}\rangle = \mathrm{n}\}$. *Then*

$$[\mathrm{Y}(\mathrm{a}, \zeta_1), \mathrm{Y}(\mathrm{b}, \zeta_2)] =$$
$$\mathrm{m}^{-1} \sum_{\mathrm{p} \in \mathrm{J}(-1)} \varepsilon_2(\nu^{\mathrm{p}}\bar{\mathrm{a}}, \bar{\mathrm{b}})\mathrm{Y}((\hat{\nu}^{\mathrm{p}}\mathrm{a})\mathrm{b}, \zeta_2)\delta(\omega^{-\mathrm{p}}\zeta_1/\zeta_2)$$
$$+ \mathrm{m}^{-2} \sum_{\mathrm{p} \in \mathrm{J}(-2)} \varepsilon_2(\nu^{\mathrm{p}}\bar{\mathrm{a}}, \bar{\mathrm{b}})[\mathrm{Y}((\hat{\nu}^{\mathrm{p}}\mathrm{a})\mathrm{b}, \zeta_2)\mathrm{D}\delta(\omega^{-\mathrm{p}}\zeta_1/\zeta_2)$$
$$+ :(\mathrm{m}\nu^{\mathrm{p}}\bar{\mathrm{a}}(\zeta_2) - \mathrm{F}(\nu^{\mathrm{p}}\bar{\mathrm{a}}, \bar{\mathrm{b}}))\mathrm{Y}((\hat{\nu}^{\mathrm{p}}\mathrm{a})\mathrm{b}, \zeta_2):\delta(\omega^{-\mathrm{p}}\zeta_1/\zeta_2)].$$

By equating the coefficients of $\zeta_1^{n_1}\zeta_2^{n_2}$ on the two sides, we obtain a formula for $[y_a(n_1), y_b(n_2)]$.

Sometimes it is useful to parametrize the vertex operators by means of elements of L rather than $\hat{L}$. To do this, choose a normalized section of $\hat{L}$—i.e., a map $\alpha \mapsto e_\alpha$ from L to $\hat{L}$ such that $e_0 = 1$ and $\overline{e_\alpha} = \alpha$ for all $\alpha \in L$. As is well known, the function ε_C: $L \times L \to \langle\omega_0\rangle$ defined by $e_\alpha e_\beta = \varepsilon_C(\alpha, \beta)e_{\alpha+\beta}$ for $\alpha, \beta \in L$ satisfies the conditions

$$\varepsilon_C(\alpha, \beta)\varepsilon_C(\alpha + \beta, \gamma) = \varepsilon_C(\beta, \gamma)\varepsilon_C(\alpha, \beta + \gamma) \quad [8.1]$$

$$\varepsilon_C(0, 0) = 1 \quad [8.2]$$

$$\varepsilon_C(\alpha, \beta)/\varepsilon_C(\beta, \alpha) = C(\alpha, \beta); \quad [8.3]$$

that is, ε_C is a normalized 2-cocycle associated with C. Conversely, any such cocycle comes from a normalized section of $\hat{L}$. Recall also that it is easy to construct such cocycles by means of bilinear functions ε_C (which automatically satisfy **[8.1]** and **[8.2]**).

Define the function η: $\mathbb{Z}/m\mathbb{Z} \times L \to \langle\omega_0\rangle$ by the condition $\hat{\nu}^p e_\alpha = \eta(p, \alpha)e_{\nu^p\alpha}$.

Also define ε: $L \times L \to \langle\omega_0\rangle$ by $\varepsilon(\alpha, \beta) = \varepsilon_2(\alpha, \beta)\varepsilon_C(\alpha, \beta)$. By **[4.6]** and **[8.3]**, ε is a normalized cocycle associated with the bilinear map $(-1)^{\langle\alpha,\beta\rangle}$.

It is easy to reformulate *Theorem 8.1* in terms of the vertex operators defined as follows for $\alpha \in L$: $Y(\alpha, \zeta) = Y(e_\alpha, \zeta) = \Sigma_{n\in\mathbb{Z}}\, y_\alpha(n)\zeta^n$. Formula **7.8** is now replaced by the following:

$$Y(\alpha, \omega^p\zeta) = \eta(p, \alpha)Y(\nu^p\alpha, \zeta) \quad [8.4]$$

for $\alpha \in L$, $p \in \mathbb{Z}/m\mathbb{Z}$.

For convenience, we assume from now on that L is *positive definite*—that is, the natural $\mathbb{R}$-linear extension of $\langle\cdot,\cdot\rangle$ to the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} L$ is positive definite. Setting $L_n = \{\alpha \in L | \langle\alpha, \alpha\rangle = n\}$ for $n \in \mathbb{Z}$, we have that $\langle\alpha, \beta\rangle = 0, \pm1, \pm2$ for $\alpha, \beta \in L_2$.

THEOREM 8.2. *Let* $\alpha, \beta \in L_2$. *Recall the notation* I(n) *from* Proposition 4.1. *We have*

$$[Y(\alpha, \zeta_1), Y(\beta, \zeta_2)] =$$

$$m^{-1} \sum_{p\in I(-1)} \eta(p, \alpha)\varepsilon(\nu^p\alpha, \beta)Y(\nu^p\alpha + \beta, \zeta_2)\delta(\omega^{-p}\zeta_1/\zeta_2)$$

$$+ m^{-2}\varepsilon(-\beta, \beta) \sum_{p\in I(-2)} \eta(p, \alpha)[D\delta(\omega^{-p}\zeta_1/\zeta_2)$$

$$- m\beta(\zeta_2)\delta(\omega^{-p}\zeta_1/\zeta_2)].$$

(We use that $\nu^p\alpha = -\beta$ for $p \in I(-2)$ and that in this case $F(\nu^p\alpha, \beta) = 0$ by **[4.5]**.)

Section 9. The Twisted Affine Algebra

Motivated by *Theorem 8.2*, we define a nonassociative algebra $(\mathfrak{g}, [\cdot,\cdot])$ over $\mathbb{F}$ as follows: $\mathfrak{g} = \mathfrak{h} \oplus \amalg_{\alpha\in L_2} \mathbb{F}x_\alpha$ ($\{x_\alpha\}_{\alpha\in L_2}$ a set of symbols), with $[\mathfrak{h}, \mathfrak{h}] = 0$, $[h, x_\alpha] = \langle h, \alpha\rangle x_\alpha = -[x_\alpha, h]$,

$$[x_\alpha, x_\beta] = \begin{cases} \varepsilon(\alpha, -\alpha)\alpha & \text{if } \alpha + \beta = 0 \\ \varepsilon(\alpha, \beta)x_{\alpha+\beta} & \text{if } \langle\alpha, \beta\rangle = -1 \\ 0 & \text{if } \langle\alpha, \beta\rangle \geq 0 \end{cases}$$

for $h \in \mathfrak{h}$, $\alpha, \beta \in L_2$. The fact that ε is a normalized cocycle associated with the bilinear map $(-1)^{\langle\alpha,\beta\rangle}$ implies that $\mathfrak{g}$ is a Lie algebra (refs. 3 and 4). We extend the form $\langle\cdot,\cdot\rangle$ from $\mathfrak{h}$ to a nonsingular symmetric form on $\mathfrak{g}$ as follows: $\langle h, x_\alpha\rangle = 0 = \langle x_\alpha, h\rangle$,

$$\langle x_\alpha, x_\beta\rangle = \begin{cases} \varepsilon(\alpha, -\alpha) & \text{if } \alpha + \beta = 0 \\ 0 & \text{if } \alpha + \beta \neq 0. \end{cases}$$

Then $\langle\cdot,\cdot\rangle$ is $\mathfrak{g}$-invariant; i.e., $\langle[x, y], z\rangle + \langle y, [x, z]\rangle = 0$ for $x, y, z \in \mathfrak{g}$. We also extend the automorphism ν of $\mathfrak{h}$ to a linear automorphism ν of $\mathfrak{g}$ by the following: $\nu x_\alpha = \eta(1, \alpha)x_{\nu\alpha}$. Then $\nu^m = 1$ on $\mathfrak{g}$,

$$\nu^p x_\alpha = \eta(p, \alpha)x_{\nu^p\alpha} \quad [9.1]$$

for $p \in \mathbb{Z}/m\mathbb{Z}$, and ν preserves $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$.

Assume for convenience that L_2 spans $\mathfrak{h}$. Then $\mathfrak{g}$ is a semisimple Lie algebra. All the conditions of section 2 of ref. 7 hold, and we can form the twisted affine Lie algebra $\tilde{\mathfrak{g}}(\nu) = \amalg_{n\in\mathbb{Z}} \mathfrak{g}_{(n)} \otimes t^n \oplus \mathbb{F}c \oplus \mathbb{F}d$ ($\mathfrak{g}_{(n)}$ denoting the ω^n-eigenspace of ν in $\mathfrak{g}$), with brackets normalized so that $[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j} + m^{-1}\langle x, y\rangle i\delta_{i+j,0}\, c$, $i, j \in \mathbb{Z}$, $x \in \mathfrak{g}_{(i)}$, $y \in \mathfrak{g}_{(j)}$ (cf. **[3.2]**). All the hypotheses of theorem 2.6 of ref. 7 hold (see **[8.4]** and *Theorem 8.2*), where in the notation of that theorem, we take $E = \mathrm{End}\, V_T$, $X'(\alpha, \zeta) = Y(\alpha, \zeta)$ for $\alpha \in L_2$, $c \mapsto 1$ and the scalar $e = m^{-1}$, and so we have the following.

THEOREM 9.1. *The representation of* $\tilde{\mathfrak{h}}(\nu)$ *on* V_T *extends uniquely to a Lie algebra representation of* $\tilde{\mathfrak{g}}(\nu)$ *on* V_T *such that* $\Sigma_{n\in\mathbb{Z}}((x_\alpha)_{(n)} \otimes t^n)\zeta^n \mapsto Y(\alpha, \zeta)$ *for all* $\alpha \in L_2$—i.e., $(x_\alpha)_{(n)} \otimes t^n \mapsto y_\alpha(n)$ *for all* $\alpha \in L_2$, $n \in \mathbb{Z}$.

Thus we have constructed the twisted affine Lie algebra by means of twisted vertex operators.

PROPOSITION 9.2. *The* $\tilde{\mathfrak{g}}(\nu)$*-module* V_T *is irreducible if and only if the* $\hat{N}$*-module* T *is irreducible* (*cf.* Sections 6 and 7).

Section 10. Shifted Operators and Their Commutators

Fix an element $\gamma \in \mathfrak{h}_{(0)}$. We shall construct "$\gamma$-shifted" analogues of the vertex operators already considered, and we shall compute certain brackets. The shifted operators should be thought of as adapted to the coset $L + \gamma$ of L.

Define a γ-shifted action of $\mathfrak{h}_{(0)}$ on U_T as follows: $\alpha^\gamma \cdot b \otimes t = \langle\alpha, \bar{b} + \gamma\rangle b \otimes t$ for $\alpha \in \mathfrak{h}_{(0)}$, $b \in \hat{L}$, $t \in T$ (cf. **[7.1]**). Correspondingly, define

$$\zeta^{\alpha^\gamma} \cdot b \otimes t = \zeta^{\langle\alpha,\bar{b}+\gamma\rangle} b \otimes t$$

(cf. **[7.2]**). Here and throughout this section, we allow arbitrary (not necessarily integral) values of $\mathbb{F}$ as exponents of formal variables such as ζ. The expected algebraic rules shall hold for such symbols.

We also define a γ-shifted $\mathbb{F}$-grading on U_T as follows: $\deg^\gamma(b \otimes t) = -(1/2)\langle\Sigma\nu^p(\bar{b} + \gamma), \bar{b} + \gamma\rangle$ (cf. **[7.4]**). Correspondingly, we have $U_T = \amalg_{n\in\mathbb{F}}(U_T)_n$, and we define a γ-shifted action d^γ of the operator d on U_T so as to act as multiplication by $\deg^\gamma(b \otimes t)$ on $b \otimes t$.

Retaining the original (trivial) action of $\tilde{\mathfrak{h}}(\nu)'$ on U_T, we obtain a new $\tilde{\mathfrak{h}}(\nu)$-module structure on U_T, and hence on V_T, by the tensor product action. The isomorphism **7.5** still holds, with $\tilde{\mathfrak{h}}(\nu)$ acting in the γ-shifted way on $\mathbb{F}[P_0L]$. We have the corresponding analogue of **[7.6]**, where the grading is now an $\mathbb{F}$-grading.

For $a \in \hat{L}$, we use formula **7.7**, with $\Sigma\nu^p\bar{a}$ replaced by $(\Sigma\nu^p\bar{a})^\gamma$, to define the γ-shifted vertex operator $X^\gamma(a, \zeta)$, and we define its components as follows: $X^\gamma(a, \zeta) = \Sigma_{n\in\mathbb{F}}\, x_a^\gamma(n)\zeta^n$; similarly, for $Y^\gamma(a, \zeta)$ ($\alpha \in L$). Then $DX^\gamma(a, \zeta) = [d^\gamma, X^\gamma(a, \zeta)]$ (cf. **[7.9]**). Also, $x_a^\gamma(n) \neq 0$ only if $n \in -m\langle\bar{a}, \gamma\rangle + \mathbb{Z}$. Since $X^\gamma(a, \zeta) = X(a, \zeta)\zeta^{-m\langle\bar{a},\gamma\rangle}$, it is easy to derive commutator formulas for γ-shifted vertex operators from *Theorems 8.1 and 8.2*.

Suppose that γ lies in the rational span of L. Choose $M \in \mathbb{N}$, $M > 0$ such that $M\langle\alpha, \gamma\rangle \in \mathbb{Z}$ for $\alpha \in L$, and let $\omega_\gamma \in \mathbb{F}$ be a primitive mMth root of unity such that $\omega_\gamma^M = \omega$. For $a \in \hat{L}$ define $Y^{\gamma,M}(\alpha, \zeta) = Y^\gamma(\alpha, \zeta^M)$. Define a linear automorphism ν_γ of $\mathfrak{g}$ by asserting that $\nu_\gamma = \nu$ on $\mathfrak{h}$, $\nu_\gamma = \nu\omega_\gamma^{-mM\langle\alpha,\gamma\rangle}$ on x_α for $\alpha \in L_2$. Then ν_γ is a Lie algebra automorphism of $\mathfrak{g}$ preserving $\langle\cdot,\cdot\rangle$, and $\nu_\gamma^{mM} = 1$ on $\mathfrak{g}$. For $p \in \mathbb{Z}/mM\mathbb{Z}$ and $\alpha \in L$, define $\eta_\gamma(p, \alpha) = \omega_\gamma^{-pmM\langle\alpha,\gamma\rangle}\eta(p, \alpha)$. Then the obvious ana-

logue of [9.1] holds for η_γ. Under the assumptions of *Theorem 8.2*, we find that $[Y^{\gamma,M}(\alpha, \zeta_1), Y^{\gamma,M}(\beta, \zeta_2)]$ equals the expression in the right-hand side in that theorem, with m replaced by mM; $I(n)$, by its analogue for $\mathbb{Z}/mM\mathbb{Z}$; η, by η_γ; Y, by $Y^{\gamma,M}$; ω, by ω_γ; and $\beta(\zeta_2)$, by $\beta(\zeta_2^M) + \langle\beta, \gamma\rangle$.

Consider the twisted affine algebra $\tilde{\mathfrak{h}}(\nu_\gamma)$, where we take the scalar e of section 2 of ref. 7 to be $(mM)^{-1}$. Then the $\tilde{\mathfrak{h}}(\nu)$-module V_T becomes an $\tilde{\mathfrak{h}}(\nu_\gamma)$-module with $c \mapsto 1$, when we use the Lie algebra isomorphism $\tilde{\mathfrak{h}}(\nu_\gamma) \to \tilde{\mathfrak{h}}(\nu)$ taking $x \otimes t^{Mn} \mapsto x \otimes t^n$ ($n \in \mathbb{Z}$, $x \in \mathfrak{h}_{(Mn)}$), $c \mapsto c$ and $d \mapsto Md$. Using theorem 2.6 of ref. 7 with $E = \mathrm{End}\, V_T$, $X'(\alpha, \zeta) = Y^{\gamma,M}(\alpha, \zeta)$ for $\alpha \in L_2$, $c \mapsto 1$ and $e = (mM)^{-1}$, we have the following generalization of the results of *Section 9*.

THEOREM 10.1. *The representation of* $\tilde{\mathfrak{h}}(\nu_\gamma)$ *on* V_T *extends uniquely to a representation of the twisted affine algebra* $\tilde{\mathfrak{g}}(\nu_\gamma)$ *on* V_T *such that* $\Sigma_{n\in\mathbb{Z}}((x_\alpha)_{(n)} \otimes t^n)\zeta^n \mapsto Y^{\gamma,M}(\alpha, \zeta)$ *for* $\alpha \in L_2$ ($(x_\alpha)_{(n)}$ *defined with respect to* ν_γ), *and* V_T *is* $\tilde{\mathfrak{g}}(\nu_\gamma)$-*irreducible if and only if* T *is* $\hat{N}$-*irreducible*.

Remark: The $\tilde{\mathfrak{g}}(\nu_\gamma)$-modules V_T in the irreducible case are basic modules (level 1 standard modules for affine algebras of types A, D, and E). For distinct pairs ν, γ, these algebras and modules can be isomorphic, and it can be very interesting to examine such isomorphisms, as in refs. 11 and 12. In particular, one gets nontrivial character identities, using [7.5]. In this connection, it is often convenient to replace t by $t^{1/mM}$ in the definition of the twisted affine algebra and ζ by $\zeta^{1/mM}$ in the definition of the vertex operator, as in formula 4 and theorem 4 of ref. 11, for example.

Note Added in Proof. *Theorem 8.1* has the following complete and surprisingly concise generalization. In the field of formal Laurent series $\Sigma_{n>N} a_n z^n$ ($a_n \in \mathbb{F}$) in a new variable z, set $y = 1 - (1 - z)^{1/m}$. Then for all $a, b \in \hat{L}$, $[Y(a, \zeta_1), Y(b, \zeta_2)]$ is the coefficient of z^{-1} in

$$m^{-1} \sum_{r\in\mathbb{Z}/m\mathbb{Z}} Y(\hat{\nu}^r a, \zeta_2(1 - y))Y(b, \zeta_2)((1 - y)^{-D-m}\delta)(\omega^{-r}\zeta_1/\zeta_2).$$

Here the product of vertex operators is to be put into "normal ordered" form using [3.4] and then expanded in powers of y and hence z. Lie algebras generated by the components of given vertex operators can be studied by iterating this formula.

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