[4] Taylor, Sir Geoffrey in *Advances in Geophysics* (New York: Academic Press, Inc., 1959), vol. 6, p. 101.

[5] Batchelor, G. K., *ibid.*, p. 449.

[6] Taylor, Sir Geoffrey, *Proc. London Math. Soc.*, **20**, 196 (1921).

[7] Obukhov, A. M., in *Advances in Geophysics* (New York: Academic Press, Inc., 1959), vol. 6, p. 113.

[8] Uhlenbeck, G. E., and L. S. Ornstein, *Phys. Rev.*, **36**, 823 (1930); Chandrasekhar, S., *Rev. Mod. Phys.*, **15**, 1 (1943).

[9] It is also the requirement in Taylor's theory of diffusion by continuous movements. Generally speaking, τ must not be so large that $\int_{\tau_1}^{\tau} R(\tau)\,d\tau$ is no longer negligible compared with $\int_0^{\tau_1} R(\tau)\,d\tau$.

[10] It is not difficult to include $v^{(0)}$ and $r^{(0)}$ in our analysis. The details have been carried out and will be presented in a later communication where other refinements and extensions will be included.

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*UNITARY OPERATOR BASES**

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To qualify as the fundamental quantum variables of a physical system, a set of operators must suffice to construct all possible quantities of that system. Such operators will therefore be identified as the generators of a complete operator basis. Unitary operator bases are the principal subject of this note.[1]

Two state vector space coordinate systems and a rule of correspondence define a unitary operator. Thus, given the two ordered sets of vectors $\langle a^k|$, $\langle b^k|$, $k = 1..N$, and their adjoints, we construct

$$U_{ab} = \sum_{k=1}^{N} |a^k\rangle\langle b^k|$$

$$U_{ba} = \sum_{k=1}^{N} |b^k\rangle\langle a^k|$$

which are such that

$$\langle a^k|U_{ab} = \langle b^k|,\ U_{ab}|b^k\rangle = |a^k\rangle$$

$$\langle b^k|U_{ba} = \langle a^k|,\ U_{ba}|a^k\rangle = |b^k\rangle$$

and

$$U_{ab}U_{ba} = U_{ba}U_{ab} = 1,\ U_{ab}\dagger = U_{ba},$$

implying the unitary property

$$U\dagger = U^{-1}$$

for both U_{ab} and U_{ba}. If a third ordered coordinate system is given, $\langle c^k|$, $k = 1..N$, we can similarly define the unitary operators U_{ac}, U_{bc}, which obey the composition property

$$U_{ab}U_{bc} = U_{ac}.$$

A unitary operator is also implied by two orthonormal operator bases in a given space, that have the same multiplication properties:

$$X(\alpha')X(\alpha'') = \sum_{\alpha=1}^{N^2} X(\alpha)\,\langle\alpha|\,\alpha'\alpha''\rangle$$

$$Y(\alpha')Y(\alpha'') = \sum_{\alpha} Y(\alpha)\,\langle\alpha|\,\alpha'\alpha''\rangle.$$

Let us define

$$U = \lambda \sum_{\alpha} X(\alpha)Y(\alpha)\dagger$$

and observe that

$$X(\alpha')U = \lambda \sum_{\alpha\alpha''} X(\alpha)\,\langle\alpha|\,\alpha'\alpha''\rangle\, Y(\alpha'')\dagger = UY(\alpha'),$$

where the latter statement follows from the remark that the $Y(\alpha)\dagger$ form an orthonormal basis and therefore

$$Y(\alpha)\dagger Y(\alpha') = \sum_{\alpha''} Y(\alpha'')\dagger tr Y(\alpha)\dagger Y(\alpha')Y(\alpha'')$$

$$= \sum_{\alpha''} \langle\alpha|\,\alpha'\alpha''\rangle\, Y(\alpha'')\dagger.$$

We also have the adjoint relation

$$Y(\alpha)\dagger U\dagger = U\dagger X(\alpha)\dagger$$

and in consequence

$$UU\dagger = \lambda\sum X(\alpha)Y(\alpha)\dagger U\dagger$$

$$= \lambda\sum X(\alpha)U\dagger X(\alpha)\dagger = 1\lambda tr U\dagger,$$

according to the completeness of the $X(\alpha)$ basis. Thus the operator U is unitary if we choose

$$\lambda^{-1} = tr U\dagger = \lambda^* \sum_{\alpha} \langle X(\alpha)|\,Y(\alpha)\rangle,$$

and, to within the arbitrariness of a phase constant,

$$\lambda = [\sum \langle X(\alpha)|\,Y(\alpha)\rangle]^{-1/2}$$

The converse theorems should be noted. For any unitary operator U, the orthonormal basis

$$Y(\alpha) = U^{-1} X(\alpha)\, U$$

obeys the same multiplication law as the $X(\alpha)$, and the $Y(\alpha)\dagger$ are given by the same linear combination of the $Y(\alpha)$ as are the $X(\alpha)\dagger$ of the $X(\alpha)$ set. In particular, if $X(\alpha)$ is a Hermitian basis, so also is $Y(\alpha)$.

We cannot refrain from illustrating these remarks for the simplest of the N^2-dimensional operator spaces, the quaternion space associated with a physical system possessing only two states. If a particular choice of these is arbitrarily designated as $+$ and $-$, we obtain the four measurement symbols $M(\pm, \pm)$, and can then introduce a Hermitian orthonormal operator basis

$$X(\alpha) = 2^{-1/2}\sigma_\alpha \quad \alpha = 0,..3$$

such that

$$\sigma_0 = 1.$$

Accordingly, the three operators σ_k, $k = 1, 2, 3$, obey

$$tr\ \sigma_k = 0, \quad {}^1/_2 tr\ \sigma_k\sigma_l = \delta_{kl},$$

and an explicit construction is given by

$$\sigma_1 = M(+, -) + M(-, +),\ \sigma_2 = -iM(+, -) + iM(-, +),$$
$$\sigma_3 = M(+, +) - M(-, -),$$

the coefficients of which constitute the well-known Pauli matrices. With these definitions, the multiplication properties of the σ operators can be expressed as

$$\sigma_k\sigma_l = \delta_{kl} + i\sum_m \epsilon_{klm}\sigma_m,$$

or, equivalently, by the additional trace

$${}^1/_2 tr \sigma_k\sigma_l\sigma_m = i\epsilon_{klm}$$

where ϵ_{klm} is the alternating symbol specified by $\epsilon_{123} = +1$. If we now introduce any other Hermitian orthonormal operator basis $Y(\alpha) = 2^{-1/2}\bar{\sigma}_\alpha$, with $\bar{\sigma}_0 = 1$, the resulting three-dimensional orthonormal basis transformation

$$\bar{\sigma}_k = \sum_{l=1}^{3} r_{kl}\sigma_l, \qquad k = 1, 2, 3$$

is real and orthogonal,

$$r^* = r,\ r^T r = 1.$$

The multiplication properties of the σ-basis assert that

$${}^1/_2\ tr\ \bar{\sigma}_k\bar{\sigma}_l\bar{\sigma}_m = i\epsilon_{klm}\ det\ r,$$

where, characteristic of an orthogonal transformation, $det\ r = \pm 1$. If the orthogonal transformation is proper, the multiplication properties of the $\bar{\sigma}$-basis coincide with those of the σ-basis, while with an improper transformation the opposite sign of i is effectively employed in evaluating the $\sigma_k\bar{\sigma}_l$ products. Hence only in the first situation, that of a pure rotation, does a unitary operator exist such that

$$\sum_l r_{kl}\sigma_l = U^{-1}\sigma_k U.$$

The unitary operator is constructed explicitly as[2]

$$U = \lambda^1/_2 \sum_{\alpha=0}^{3} \sigma_\alpha\bar{\sigma}_\alpha$$
$$= \lambda^1/_2[1 + tr\ r + i\sum_{m=1}^{3} r_{kl}\epsilon_{klm}\sigma_m]$$

with

$$\lambda = (1 + tr\ r)^{-1/2}.$$

Let us return to the definition of a unitary operator through the mapping of one coordinate system or another, and remark that the two sets of vectors can be identical, apart from their ordering. Thus, consider the definition of a unitary operator V by the cyclic permutation

$$\langle a^k | V = \langle a^{k+1} |, \; k = 1..N$$

where

$$\langle a^{N+1} | = \langle a^1 |,$$

which indicates the utility of designating the same state by any of the integers that are congruent with respect to the modulus N. The repetition of V defines linearly independent unitary operators,

$$\langle a^k | V^n = \langle a^{k+n} |,$$

until we arrive at

$$\langle a^k | V^N = \langle a^{k+N} | = \langle a^k |.$$

Thus

$$V^N = 1$$

is the minimum equation, the polynomial equation of least degree obeyed by this operator, which we characterize as being of period N. The eigenvalues of V obey the same equation and are given by the N distinct complex numbers

$$v' = e^{\frac{2\pi i k}{N}} = v^k, \qquad k = 0, ..N - 1.$$

Unitary operators can be regarded as complex functions of Hermitian operators, and the entire spectral theory of Hermitian operators can be transferred to them. If the unitary operator V has N distinct eigenvalues, its eigenvectors constitute an orthonormal coordinate system. The adjoint of a right eigenvector $| v' \rangle$ is the left eigenvector $\langle v' |$ associated with the same eigenvalue, and the products

$$| v' \rangle \langle v' | = M(v')$$

have all the properties required of measurement symbols. Now let us observe that the factorization of the minimum equation for V that is given by

$$[(V/v') - 1] \sum_{l=0}^{N-1} (V/v')^l = 0$$

permits the identification of the Hermitian operator

$$M(v') = 1/N \sum (v')^{-l} V^l$$

to within the choice of the factor N^{-1}, which is such that

$$\langle v' | M(v') = \langle v' |.$$

On multiplying $M(v^k)$ by $\langle a^N |$ and using the defining property of V, we obtain

$$\langle a^N | v^k \rangle \langle v^k | = 1/N \sum_l \langle a^l | e^{-\frac{2\pi i}{N} kl}$$

from which follows

$$|\langle a^N | v^k \rangle|^2 = 1/N.$$

Then, with a convenient phase convention for $\langle a^N | v^k \rangle$ we get

$$\langle v^k | = N^{-1/2} \sum_l \langle a^l | e^{-\frac{2\pi i}{N} kl}$$

which is also expressed by

$$\langle v^k | a^l \rangle = N^{-1/2} e^{-\frac{2\pi i}{N} kl}, \ \langle a^l | v^k \rangle = N^{-1/2} e^{\frac{2\pi i}{N} kl},$$

the elements of the transformation functions that connect the given coordinate system with the one supplied by the eigenvectors of the unitary operator that cyclically permutes the vectors of the given system.

Turning to the new coordinate system $\langle v^k |$, we define another unitary operator by the cyclic permutation of this set. It is convenient to introduce U such that

$$\langle v^k | U^{-1} = \langle v^{k+1} |$$

which is equivalent to

$$\langle v^k | U = \langle v^{k-1} |.$$

This operator is also of period N,

$$U^N = 1,$$

and has the same spectrum as V,

$$u' = e^{\frac{2\pi i k}{N}} = u^k, \ k = 0..N - 1.$$

After using the property $U^{N-l} = U^{-l}$ to write the corresponding measurement symbol as

$$M(u') = 1/N \sum_{l=0}^{N-1} (u')^l U^{-l},$$

we follow the previous procedure to construct the eigenvectors of U,

$$\langle u^k | = N^{-1/2} \sum_l \langle v^l | e^{\frac{2\pi i}{N} kl} = \sum_l \langle a^k | v^l \rangle \langle v^l | = \langle a^k |.$$

Thus, the original coordinate system is regained and our results can now be stated as the reciprocal definition of two unitary operators and their eigenvectors,

$$\langle u^k | V = \langle u^{k+1} |$$

$$\langle v^k | U^{-1} = \langle v^{k+1} |.$$

The relation between the two coordinate systems is given by

$$\langle u^k | v^l \rangle = N^{-1/2} e^{\frac{2\pi i}{N} kl}, \ \langle v^l | u^k \rangle = N^{-1/2} e^{-\frac{2\pi i}{N} kl},$$

and, supplementing, the periodic properties

$$U^N = V^N = 1,$$

we infer from the comparison

$$\langle u^k | UV = \langle u^{k+1} | u^k, \ \langle u^k | VU = \langle u^{k+1} | u^{k+1} = \langle u^{k+1} | u^k e^{\frac{2\pi i}{N}}$$

that

$$VU = e^{\frac{2\pi i}{N}} UV.$$

As a consequence of the latter result, we also have

$$V^l U^k = e^{\frac{2\pi i}{N} kl} U^k V^l.$$

Each of the unitary operators U and V is a function of a Hermitian operator that in itself forms a complete set of physical properties. It is natural to transfer this identification directly to the unitary operators which are more accessible than the implicit Hermitian operators. Accordingly, we now speak of the statistical relation between the properties U and V, as described by the probability

$$p(u', v'') = |\langle u' | v'' \rangle|^2 = 1/N.$$

The significance of this result can be emphasized by considering a measurement sequence that includes a nonselective measurement, as in

$$p(u', v, u'') = \sum_{v'} p(u', v')p(v', u'') = 1/N,$$

for this asserts that the intervening non-selective v-measurement has destroyed all prior knowledge concerning u-states. Thus the properties U and V exhibit the maximum degree of incompatibility. We shall also show that U and V are the generators of a complete orthonormal operator basis, such as the set of N^2 operators

$$X(mn) = N^{-1/2} U^m V^n, \ m, n = 0..N - 1,$$

and therefore together supply the foundation for a full description of a physical system possessing N states. Both of these aspects are implied in speaking of U and V as a *complementary* pair of operators.[3] Incidentally, there is complete symmetry between U and V, as expressed by the invariance of all properties under the substitution

$$U \rightarrow V, \ V \rightarrow U^{-1}.$$

The latter could be emphasized by choosing the elements of the operator basis as

$$N^{-1/2} e^{\frac{\pi i}{N} mn} U^m V^n = N^{-1/2} e^{-\frac{\pi i}{N} mn} V^n U^m,$$

which are invariant under this substitution when combined with $m \rightarrow n, n \rightarrow -m$.

One proof of completeness for the operator basis generated by U and V depends upon the following lemma: If an operator commutes with both U and V it is necessarily a multiple of the unit operator. Since U is complete in itself, such an operator must be a function of U. Then, according to the hypothesis of commutativity with V, for each k we have

$$0 = \langle u^k | [f(U)V - Vf(U)] | u^{k+1} \rangle = f(u^k) - f(u^{k+1}),$$

and this function of U assumes the same value for every state, which identifies it with that multiple of the unit operator. Now consider, for arbitrary Y,

$$\sum_{mn} X(mn)YX(mn)\dagger = 1/N \sum_{mn} U^m V^n Y V^{-n} U^{-m} = 1/N \sum_{mn} V^n U^m Y U^{-m} V^{-n},$$

and observe that left and right hand multiplication with U and U^{-1}, respectively, or with V and V^{-1}, only produces a rearrangement of the summations. Accordingly, this operator commutes with U and V. On taking the trace of the resulting equation, the multiple of unity is identified with $tr\ Y$, and we have obtained

$$\sum_{m,\, n\, =\, 0}^{N-1} X(mn)YX(mn)\dagger = 1\ tr\ Y,$$

the statement of completeness for the N^2-dimensional operator basis $X(mn)$. Alternatively, we demonstrate that these N^2 operators are orthonormal by evaluating

$$\langle X(mn) | X(m'n')\rangle = 1/N\ tr\ U^{m'-m} V^{n'-n}$$
$$= \delta(m, m')\delta(n,n'),\ m, n, m', n' = 0..N - 1.$$

The unit value for $m = m'$, $n = n'$ is evident. If $m \neq m'$, the difference $m' - m$ can assume any value between $N - 1$ and $-(N - 1)$, other than zero. When the trace is computed in the v-representation, the operator $U^{m'-m}$ changes each vector $\langle v^k|$ into the orthogonal vector $\langle v^{k+m-m'}|$ and the trace vanishes. Similarly, if $n \neq n'$ and the trace is computed in the u-representation, each vector $\langle u^k|$ is converted by $V^{n'-n}$ into the orthogonal vector $\langle u^{k+n'-n}|$ and the trace equals zero.

One application of the operator completeness property is worthy of attention. We first observe that

$$U^m V^n U V^{-n} U^{-m} = e^{\frac{2\pi i}{N} n}\ U = u^n U$$
$$U^m V^n V V^{-n} U^{-m} = e^{-\frac{2\pi i}{N} m}\ V = v^{-m} V,$$

which exhibits the unitary transformations that produce only cyclic spectral translations. Now, if Y is given as an arbitrary function of U and V, the completeness expression of the operator basis reads

$$1/N^2 \sum_{u'v'} F(u'U, v'V) = 1/N\ tr\ F$$

which is a kind of ergodic theorem, for it equates an average over all spectral translations to an average over all states. The explicit reference to operators can be removed if $F(U, V)$ is constructed from terms which, like the individual operators $X(mn)$, are ordered with U standing everywhere to the left of V. Then we can evaluate a matrix element of the operator equation, corresponding to the states $\langle u^0|$ and $|v^0\rangle$, which gives the numerical relation

$$tr\ F(U,V) = 1/N \sum_{u'v'} F(u', v').$$

It is interesting to notice that a number of the powers U^k, $k = 1 .. N - 1$, can have the period N. This will occur whenever the integers k and N have no common factor and thus the multiplicity of such operators equals $\phi(N)$, the number of integers less than and relatively prime to N. Furthermore, to every such power of U there can be associated a power V^l, also of period N, that obeys with U^k the same operator equation satisfied by U and V,

$$V^l U^k = e^{\frac{2\pi i}{N}} U^k V^l.$$

This requires the relation

$$kl = 1 \ (mod\ N),$$

and the unique solution provided by the Fermat-Euler theorem is

$$l = k^{\phi(N) - 1} \ (mod\ N).$$

The pair of operators U^k, V^l also generate the operator basis $X(mn)$, in some permuted order.

We shall now proceed to replace the single pair of complementary operators U, V by several such pairs, the individual members of which have smaller periods than the arbitrary integer N. This leads to a classification of quantum degrees of freedom in relation to the various irreducible, prime periods. Let

$$N = N_1 N_2$$

where the integers N_1 and N_2 are relatively prime, and define

$$U_1 = U^{N_2} \qquad U_2 = U^{N_1}$$
$$V_1 = V^{l_1 N_2} \qquad V_2 = V^{l_2 N_1}$$

with

$$l_1 = N_2^{\phi(N_1) - 1} \ (mod\ N_1),\ l_2 = N_1^{\phi(N_2) - 1} \ (mod\ N_2).$$

It is seen that U_1, V_1 are of period N_1, while U_2, V_2 have the period N_2, and that the two pairs of operators are mutually commutative, as illustrated by

$$V_1 U_2 = e^{\frac{2\pi i}{N} l_1 N_1 N_2} U_2 V_1 = U_2 V_1.$$

Furthermore,

$$V_1 U_1 = e^{\frac{2\pi i}{N_1}} U_1 V_1$$
$$V_2 U_2 = e^{\frac{2\pi i}{N_2}} U_2 V_2,$$

so that U_1, V_1 and U_2, V_2 constitute two independent pairs of complementary operators associated with the respective periods N_1 and N_2. We also observe that the $N = N_1 N_2$ independent powers of U can be obtained as

$$U_1^{m_1} U_2^{m_2} = U^{(m_1 N_1 + m_2 N_2)} \qquad \begin{array}{l} m_1 = 0..N_1 - 1 \\ m_2 = 0..N_2 - 1 \end{array}$$

since all of these are distinct powers owing to the relatively prime nature of N_1 and N_2. With a similar treatment for V, we recognize that the members of the orthonormal operator basis are given in some order by

$$X(m_1m_2n_1n_2) = N^{-1/2}U_1^{m_1}\,U_2^{m_2}\,V_1^{n_1}\,V_2^{n_2} = \prod_{j=1}^{2} X(m_jn_j),$$

where

$$X(m_jn_j) = N_j^{-1/2}U_j^{m_j}\,V_j^{n_j},\ m_j,\ n_j = 0..N_j - 1.$$

Another approach to this commutative factorization of the operator basis proceeds through the construction of the eigenvalue index $k = 0..N - 1$ with the aid of a pair of integers,

$$k = k_1N_2 + k_2 \qquad k_1 = 0..N_1 - 1$$
$$k_2 = 0..N_2 - 1.$$

Equivalently, we replace

$$u^k = v^k = e^{\frac{2\pi ik}{N}}$$

by

$$u_1^{k_1} = v_1^{k_1} = e^{\frac{2\pi ik_1}{N_1}}$$
$$u_2^{k_2} = v_2^{k_2} = e^{\frac{2\pi i\,k_2}{N_2}}$$

which gives

$$\langle u^k| = \langle u_1^{k_1}\,u_2^{k_2}|,\ \langle v^k| = \langle v_1^{k_1}\,v_2^{k_2}|.$$

On identifying these vectors as the eigenvectors of sets of two commutative unitary operators, we can define $U_{1,2}$, $V_{1,2}$ by the reciprocal relations

$$\langle u_1^{k_1}\,u_2^{k_2}|\,V_1 = \langle u_1^{k_1+1}\,u_2^{k_2}|,\ \langle u_1^{k_1}\,u_2^{k_2}|\,V_2 = \langle u_1^{k_1}\,u_2^{k_2+1}|$$
$$\langle v_1^{k_1}\,v_2^{k_2}|\,U_1^{-1} = \langle v_1^{k_1+1}\,v_2^{k_2}|,\ \langle v_1^{k_1}\,v_2^{k_2}|\,U_2^{-1} = \langle v_1^{k_1}\,v_2^{k_2+1}|$$

which reproduce the properties

$$U_j^{N_j} = V_j^{N_j} = 1,\ V_j\,U_j = e^{\frac{2\pi i}{N_j}}\,U_jV_j,$$

together with the commutativity of any two operators carrying different subscripts. The orthonormality of the $N_1^2N_2^2 = N^2$ operators $X(m_1m_2n_1n_2)$ can now be directly verified. Also, by an appropriate extension of the preceeding discussion, we obtain the transformation function

$$\langle u_1^{k_1}\,u_2^{k_2}|\,v_1^{l_1}\,v_2^{l_2}\rangle = \prod_{j=1}^{2} \langle u_j^{k_j}|\,v_j^{l_j}\rangle,$$

with

$$\langle u_j^{k_j}|\,v_j^{l_j}\rangle = N_j^{-1/2}e^{\frac{2\pi i}{N_j}k_j\,l_j}.$$

The continuation of the factorization terminates in

$$N = \prod_{j=1}^{f} \nu_j,$$

where f is the total number of prime factors in N, including repetitions. We call this characteristic property of N the number of degrees of freedom for a system possessing N states. The resulting commutatively factored basis

$$X(mn) = \prod_{j=1}^{f} X(m_j n_j)$$

$$X(m_j n_j) = \nu_j^{-1/2} U_j^{m_j} V_j^{n_j};\ m_j, n_j = 0 \ldots \nu_j - 1$$

is thus constructed from the operator bases individually associated with the f degrees of freedom, and the pair of irreducible complementary quantities of each degree of freedom is classified by the value of the prime integer $\nu = 2, 3, 5, \ldots \infty$. In particular, for $\nu = 2$ the complementary operators U and V are anticommutative and of unit square. Hence, they can be identified with σ_1 and σ_2, for example, and the operator basis is completed by the product $-iUV = \sigma_3$.

The characteristics of a degree of freedom exhibiting an infinite number of states can be investigated by making explicit the Hermitian operators upon which U and V depend,

$$U = e^{i\epsilon q},\ V = e^{i\epsilon p},\ \epsilon = (2\pi/\nu)^{1/2},$$

and where

$$q^k = p^k = k\epsilon, \quad k = 0, \pm 1, \pm 2, \ldots \pm {}^1/_2(\nu - 1).$$

We shall not carry out the necessary operations for $\nu \to \infty$, which evidently yield the well-known pair of complementary properties with continuous spectra. One remark must be made, however. In this approach one does not encounter the somewhat awkward situation in which the introduction of continuous spectra requires the construction of a new formalism, be it expressed in the language of Dirac's delta function, or of distributions. Rather, we are presented with the direct problem of finding the nature of the subspaces of physically meaningful states and operators for which the limit $\nu \to \infty$ can be performed uniformly.

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[1] For the notation and concepts used here see these PROCEEDINGS, **45**, 1542 (1959), and **46**, 257 (1960).

[2] The absence in the available literature of an explicit statement of this simple, general result is rather surprising. The inverse calculation giving the three-dimensional rotation matrix in terms of the elements of the unitary matrix is very well known (rotation parametrizations of Euler, Cayley-Klein), and the construction of the unitary matrix with the aid of Eulerian angles is also quite familiar.

[3] Operators having the algebraic properties of U and V have long been known from the work of Weyl, H., *Theory of Groups and Quantum Mechanics* (New York: E. P. Dutton Co., 1932), chap. 4, sect. 14, but what has been lacking is an appreciation of these operators as generators of a complete operator basis for any N, and of their optimum incompatibility, as summarized in the attribute of complementarity. Nor has it been clearly recognized that an *a priori* classification of all possible types of physical degrees of freedom emerges from these considerations.