first factor on the right in (15) yields that (15), and therefore the integrand in (12), is at least

$$\left(\frac{\alpha x\cdot x^{-1}B^{-1} + \beta y . y^{-1}A^{-1}}{\alpha x + \beta y}\right)^{-1}\cdot\frac{1}{\alpha x + \beta y} = \frac{AB}{\alpha A + \beta B}$$

which proves Theorem I.

Conditions for the equality sign in (2) are important for several applications. Because of the continuity of the functions involved, the equality sign in (2) requires the equality sign in (11) and in the two cases of Jensen's Inequality. Moreover $\bar{C}_{u(\rho)}$ must coincide with $\bar{D} \cap H_\rho$.

The equality sign holds in (11) only if the sets $\bar{A}_{x(\rho)}$ and $\bar{B}_{y(\rho)}$ are homothetic (see B. F., pp. 72 and 88). The condition $a_1 = a_2$ for equality in (14) yields in the two cases

$$A_x/A^{1-\epsilon}y^{1-\epsilon} = B_y/B^{1-\epsilon}x^{1-\epsilon} \text{ and } xB = yA$$

so that $A_x = B_y$.

Therefore $\bar{A}_x$ and $x\bar{B}_y$ are not only homothetic but congruent. The relation $x:A = y:B$ shows then that $\bar{A} = H_r \cap K$ can be transformed into $\bar{B} = H_s \cap K$ by revolving $\bar{A}$ about L through $\alpha' + \beta'$ and dilating it at the ratio $B:A$ in the direction of $\mathfrak{f}$. If w' is the point of $\bar{B}$ into which the point w of $\bar{A}$ is mapped under this transformation, it is easily seen that the segments connecting w to w' form a convex set E when w tranverses $\bar{A}$, so that E is the convex closure of $\bar{A} \cup \bar{B}$. Clearly the equality sign holds in (2) for $K = E$. Thus we find

III. *The equality sign holds in (2) if and only if $H_r \cap K$ can be transformed into $H_s \cap K$ by a rotation about L and a subsequent dilation in the direction of $\mathfrak{f}$, and $H_q \cap K$ is the intersection of H_q with the convex closure of $H_r \cap K$ and $H_s \cap K$.*

From III and the fact that E is the union of the segments connecting w and w' we obtain the following addition to II:

IIa. *The surface S in II is strictly convex if K is strictly convex.*

But III shows that the strict convexity of K is not a necessary condition.

THE n-ALITY THEORY OF RINGS

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1. *Introduction.*—The classic duality-symmetry which is exhibited by Boolean rings (and Boolean algebras), far from being characteristic of

this class, is actually a phenomenon inherent (though usually dormant) in all rings. This ring duality has been presented and variously explored in a series of papers.[1–7]

The present communication is intended as a brief partial résumé dealing with extensions of this duality theory to *n-ality,* and more generally to *K-ality* theories, corresponding to various groups K of admissible "co-ordinate transformations." The specialization $K = C =$ complementation group, of order 2 (see §3), yields the original ring duality, which latter we now designate as *simple,* or *mod K,* to distinguish it from rival theories.

The elevation of the simple theory to the general K-level, and a proper refinement thereof, throws new light on certain previous results on the C-level, particularly in connection with questions dealing with the (simple) logical-algebra definability of rings. However, the most important current results on the K-level center around the new concept, *ring-logic* (K). Roughly the latter is defined as a ring which both uniquely (equationally) determines and is uniquely (equationally) determined by its K-logical-algebra. On the simple level, for instance, a Boolean ring is a ring-logic (C).

We introduce *p-rings,* a natural generalization of Boolean rings (which are coextensive with 2-rings), formulate their *p-ality theory,* and establish that 3-rings form a new class of ring-logics. These 3-ring-logics subsume the familiar 3-valued-logic; moreover, the relationship between the latter, general 3-rings, their logical algebras and the enveloping *tri-ality theorem* on the one hand, forms an exact generalization of the relationship between 2-valued logic (= logic of propositions), general Boolean rings (= 2-rings), their logical algebras and the encompassing duality theory on the other.

2. *Fundamentals of the Simple Duality Theory.*—If $(R, +, \times)$ is a ring (with unit), the concepts of R occur in dual pairs. For example, 0 and 1 are dual elements; $\times, \times'$; $+, +'$; $-, -'$; $*$, are respective dual pairs of operations, the latter being self-dual, where

$$\begin{array}{ll}
\left.\begin{array}{l} a \times' b = a + b - (a \times b) \\ a \times b = a +' b -' (a \times' b) \end{array}\right\} & \text{dual ring products} \\
\left.\begin{array}{l} a +' b = a + b - 1 \\ a + b = a +' b -' 0 \end{array}\right\} & \text{dual ring sums} \\
\left.\begin{array}{l} a -' b = a - b + 1 \\ a - b = a -' b +' 0 \end{array}\right\} & \text{dual ring differences} \\
a^* = 1 - a = 0 -' a & \text{(self-dual) ring complement}
\end{array} \qquad (2.1)$$

Restricted for brevity to these concepts one has[1, 5] the

(SIMPLE) RING DUALITY THEOREM. *If* $P(0, 1;\ \times, \times';\ +, +';\ -, -';\ *)$ *is a true proposition of a ring* $(R, +, \times)$, *then so also is its dual*

$$dl.\, P = P(1, 0;\ \times', \times;\ +', +;\ -', -;\ *)$$

*obtained by replacing each argument by its dual, as indicated, with * left unchanged (self-dual).*

Illustrations of the duality theorem in an arbitrary ring are given by the various dual relations (2.1). Again, by

$$\left.\begin{aligned}(a \times b)^* &= a^* \times' b^* \\ (a \times' b)^* &= a^* \times b^*\end{aligned}\right\} \text{ ring "De-Morgan" formulas.} \qquad (2.2)$$

$$a^{**} = a, \quad 0^* = 1, \qquad 1^* = 0 \qquad (2.3)$$

$(R, +, \times)$ is a ring with 0 as zero element and 1 as unit.
$(R, +', \times')$ is a ring with 1 as zero element and 0 as unit. (2.4)

As additional illustrations, in a Boolean ring (or, more generally, in a Boolean-like ring,[1]) one has

$$\begin{aligned}a + b &= (a \times b^*) \times' (a^* \times b) \\ a +' b &= (a \times' b^*) \times (a^* \times' b)\end{aligned} \qquad (2.5)$$

Again, in a field, for instance,

$$a + b: \begin{cases} a + b = a \times' b \times' (a^0 \times b^0) & (a \neq 1, b \neq 1) \\ a + 1 = 1 + a = a \times' a^{10} & (a \neq 0, \neq 1) \\ 1 + 1 = a \times' a^{10} \times' a^{01} & (a \neq 0, \neq 1) \\ 1 + 0 = 0 + 1 = 1 & \end{cases} \qquad (2.6)$$

$$a +' b: \begin{cases} a +' b = a \times b \times (a^1 \times' b^1) & (a \neq 0, b \neq 0) \\ a +' 0 = 0 +' a = a \times a^{01} & (a \neq 1, \neq 0) \\ 0 + 0 = a \times a^{01} \times a^{10} & (a \neq 1, \neq 0) \\ 0 +' 1 = 1 +' 0 = 0 & \end{cases}$$

(Here a^1 and a^0 are the $\times$ and $\times'$ inverses of a, respectively.)

If R is a Boolean ring, the dual products $\times$, $\times'$ reduce to the logical product, $\cap$, and logical sum, $\cup$, respectively; * to the Boolean complement, $^-$, and duality and "De-Morgan" theorems to the corresponding Boolean theorems.

With this latter specialization as motivation, in an arbitrary ring the (operationally closed) system $(R, \times, \times', {}^*)$ was introduced in reference 1 as the (simple) *logical algebra*, or briefly the *logic* of the ring. This is an illustration of a "mixed" concept; a ring in either of the forms $(R, +, \times)$ or $(R, +', \times')$ is "pure," that is, formulated within a single "coordinate system," but "mixed" in the form $(R, +, \times, +', \times', {}^*)$. A (simple) logical concept of a ring is one definable in terms of the logical-algebra of the ring; a (simply) *logically definable ring* is one whose ring sum, +, and therefore the entire ring, is a simple logical (= C-logical) concept. It has been shown that Boolean rings,[1, 3, 5] Boolean-like rings,[1] fields,[6] and integral domains are examples of C-logically definable rings.

3. *General Transformation Theory.*—If $U = \{\ldots, x, \ldots\}$ is a class

(with or without structure), $\varphi(x, y, \ldots)$ any multitation of (= operation of one or more arguments in) U, and $\rho(x)$ any permutation (= 1-1 monotation) of U, with inverse written ρ^-, then in the "ρ coördinate system" the "point" x receives the new "coördinate" $\rho(x)$, and the multitation φ "becomes" φ_ρ, where

$$\varphi_\rho = \rho^-(\varphi(\rho(x), \rho(y), \ldots). \tag{3.1}$$

Based on this simple extension of the "transform" concept, the traditional (*a*) transformation and invariant theories (various groups, tensors, matrices, etc.), in all of which the underlying (*b*) "computational" disciplines (analysis, arithmetic, a particular ring—in fact any operational algebra) are regarded as absolute invariants (unchanged by coördinate transformations), may be brought under a single unifying theory in which a given such "computational" discipline may itself profitably be thought of as transforming "cogrediently," or sometimes "contragrediently" with changing "coördinates."

We consider here only rings. If $(R, +, \times)$ is a ring and K a preassigned group of coördinate transformations in (= permutations of) the set R, then in the ρ coördinate system the basic ring operations become

$$\begin{aligned} a + b \rightarrow a +_\rho b &= \rho^-(\rho(a) + \rho(b)) \\ a \times b \rightarrow a \times_\rho b &= \rho^-(\rho(a) \times \rho(b)) \end{aligned} \tag{3.2}$$

If $+, +', +'', \ldots$ and $\times, \times', \times'', \ldots$ are the (conjugate) classes $\{+_\rho\}$ and $\{\times_\rho\}$ of all transforms (3.1) of $+$ and of $\times$ by the various $\rho \in K$, then $(R, +, \times)$, $(R, +', \times')$, ..., etc., denote the "same" ring, in different coördinates, precisely as in the (subsumed special) case of tensors, matrices, etc., in different coördinates.

If, in particular, the admissible group K of coördinate transformations is chosen as C (order 2),

$$x^* = 1 - x, \quad x^{**} = x = \text{identity}, \tag{3.3}$$

the C-ality ring theory reduces to the original *simple* theory. Further specialized to a Boolean ring, for instance, the duality of logical-sum $\cup$ (= $\times'$) and logical-product $\cap$ (= $\times$) is simply a way of saying that these are *identical* notions, expressed in different coördinates.

For general K the *K-logical algebra* (= *K-logic*) of a ring $(R, +, \times)$ is defined as the (operationally closed) system

$$(R, \times, \times', \times'', \ldots, \xi, \rho, \rho', \rho'', \ldots) \tag{3.4}$$

whose operations are the various monotations ξ (= identity), ρ, ρ', ρ'', ... of K, and the (conjugate set of bitations) $\times, \times', \times''$, ... of transforms (3.1) of $\times$ by the ξ, ρ, ρ', ... of K.

A ring is: (*a*) *K-logically definable* if its ring sum, $+$, (and hence the whole ring) is a K-logical concept; (*b*) K-logically *equationally* definable if its $+$ is compositionally generable by the K-logical operations $\times$, $\times'$, $\times''$, $\ldots$, ξ, ρ, ρ', $\ldots$, or, what is equivalent, if its $+$ satisfies an identity

$$a + b \underset{a,\, b}{\equiv} \varphi(a, b) \tag{3.5}$$

where φ is compositionally generable by $a \times b$, $\xi(a)(= a)$, $\rho(a)$, $\rho'(a) \ldots$; (*c*) K-logically *fixed* if it is K-logically definable, and if no ring $(R, +_1, \times)$ exists (on the same class R and with the same $\times$, but with different ring sum $+_1 \neq +$), which is K-logically definable.

On the C-level the following results are established:

THEOREM I. *(1°) A Boolean ring is C-logically fixed and equationally definable. (2°) A Boolean-like ring is C-logically definable but not in general C-logically fixed. (3°) A field is C-logically fixed, but not in general C-logically equationally definable.*

We introduce the concept *ring-logic* (K) as synonymous with a ring which is both K-logically fixed and equationally definable. The above Theorem asserts that a Boolean ring is a ring-logic (C), while a field, or a Boolean-like ring is generally not. In a ring-logic (K) we capture in generalized form the extreme intimacy between ring and logical algebra which, on the C-level, is exemplified by the class of Boolean ring-logics. The generalized parallelism is still closer when the K-ality theory which permeates ring and logic is recognized.

4. *p-Rings.*—We arrive at a new class of ring-logics by considering *p-rings*, a natural generalization of Boolean rings. A p-ring (p = prime integer) is a commutative ring with unit $(S, +, \times)$, in which for all $a \in S$

$$a^p = a \times a \times \ldots \times a = a \tag{4.1}$$

$$pa = a + a + \ldots + a = 0 \tag{4.2}$$

Thus the concepts 2-ring and Boolean ring coincide; for $p = 2$, (4.2) follows from (4.1), but not for $p > 2$. For given p, the simplest p-ring is $(F_p, + \times)$, the field of residues mod p. As in the special Boolean case, all finite p-rings are direct products

$$F_p \times F_p \times F_p \times \ldots \times F_p \tag{4.3}$$

of the prime field F_p, and hence have p^t elements. We here omit considerations of general ideal structure.

The "cyclic negation group" N (of order p) of a p-ring is the cyclic group generated by $x^\wedge$,

$$x^\wedge = 1 + x \tag{4.4}$$

(For 2-rings N is seen to be identical with C, since $x^\wedge = 1 + x = 1 - x = x^*$ in a Boolean ring.)

Since we here consider only the above group N of coördinate transformations in a p-ring, we may omit reference to it. Thus, p-ality $\longleftrightarrow$ p-ality (N), etc.

Each p-ring has a *p-ality theory*, that is, each concept or proposition of the ring is one of a p-al set of p such concepts (or is self p-al, i.e., the same in all admissible coördinates). For convenience, however, we shall deal with the *tri-ality* case ($p = 3$), from which the form of the general p-ality theorem will at once be clear. For $p = 3$ we abbreviate $x^{\wedge\wedge} = x^{\vee}$, and hence

$$\begin{aligned} x^{\wedge\vee} &= x^{\vee\wedge} = x^{\wedge\wedge\wedge} = x^{\vee\vee\vee} = x \\ x^{\wedge\wedge} &= x^{\vee}, x^{\vee\vee} = x^{\wedge}. \end{aligned} \tag{4.5}$$

From (3.1) and (4.4) the tri-als if the ring product $\times$ are given by:

$$\begin{aligned} a \times' b &= (a^{\wedge} \times b^{\wedge})^{\vee} = a \times b + a + b \\ a \times'' b &= (a^{\vee} \times b^{\vee})^{\wedge} = a \times b + 2(a + b) + 2, \end{aligned} \tag{4.6}$$

and similarly

$$\begin{aligned} a +' b &= (a^{\wedge} + b^{\wedge})^{\vee} = a + b + 1 \\ a +'' b &= (a^{\vee} + b^{\vee})^{\wedge} = a + b + 2 \\ a -' b &= (a^{\wedge} - b^{\wedge})^{\vee} = a - b - 1 \\ a -'' b &= (a^{\vee} - b^{\vee})^{\wedge} = a - b - 2 \end{aligned} \tag{4.7}$$

In a formula a "multiplicative" constant, such as the 2 in 2 $(a + b)$ is an *apparent* (or *removable*) constant,

$$2(a + b) = a + b + a + b$$

An "additive" constant is *real.*

THEOREM II. *Tri-ality Theorem for 3-Rings.*

Let

$$P(0, 1, 2;\ \times, \times', \times'';\ +, +', +'';\ -, -', -'';\ {}^{\wedge};\ {}^{\vee})$$

be any true proposition in a 3-ring, S, where P involves no apparent constants. Then each of the propositions tri-al to P,

$$\begin{aligned} P' &= P(2,0,1;\ \times', \times'', \times;\ +', +'', +;\ -', -'', -;\ {}^{\wedge};\ {}^{\vee}) \\ P'' &= P(1,2,0;\ \times'', \times, \times';\ +'', +, +';\ -'', -, -';\ {}^{\wedge};\ {}^{\vee}) \end{aligned} \tag{4.8}$$

obtained by (a) leaving each of the operations $^{\wedge}, {}^{\vee}$ unchanged, (b) applying any cyclic permutation "cograediently" to all other operations, and (c) the "contragredient" (= inverse) permutation to the real constants, is again a true proposition of S.

The "contragredient" element is not apparent in the duality case $p = 2$, since in this special case $x^{\wedge} = x^{\vee} = x^{*}$ is its own inverse. The self-tri-ality of $^{\wedge}$ as well as $^{\vee}$ is similar to the situation for $*$ in the Boolean case.

Of the multitude of illustrations of the tri-ality theorem in any 3-ring we can here mention only two of the simplest.

THEOREM III. *Transformation Theorem (or "De Morgan" Formulas) for 3-Rings. In any 3-ring,*

$$\begin{aligned} a \times' b &= (a^\wedge \times b^\wedge)^\vee = (a^\vee \times'' b^\vee)^\wedge \\ a \times'' b &= (a^\vee \times b^\vee)^\wedge = (a^\wedge \times' b^\wedge)^\vee \\ a \times b &= (a^\wedge \times'' b^\wedge)^\vee = (a^\vee \times' b^\vee)^\wedge \end{aligned} \tag{4.9}$$

These are the equations of "conversion" from each permissible coördinate system to any other such. Similarly the conversion formulas for the tri-al ring sums and differences are given by

THEOREM IV. *In any 3-ring,*

$$\begin{aligned} a +' b &= a + b + 1 = a +'' b +'' 0 \\ a +'' b &= a + b + 2 = a +' b +' 0 \\ a + b &= a +'' b +'' 2 = a +' b +' 1 \\ a -' b &= a - b - 1 = a -'' b -'' 0 \\ a -'' b &= a - b - 2 = a -' b -' 0 \\ a - b &= a -'' b -'' 2 = a -' b -' 1 \end{aligned} \tag{4.10}$$

5. *3-Ring-Logics.*—For 3-rings, always referring to the cyclic negation group N (of order 3), we have established

THEOREM V. *Each 3-ring* $(S, +, \times)$ *is a ring-logic, with* $+$ *logically definable by the equation*

$$a + b = ab^\wedge \times' a^\wedge b \times' a^2 b^2 \tag{5.1}$$

This theorem may readily be tri-alized.

Since $N = C$ for Boolean rings ($=$ 2 rings), if one compares (5.1) with the corresponding formula for 2-rings, namely

$$a + b = ab^\wedge \times' a^\wedge b$$

it is found that neither "covers" the other, that is, neither reduces to the correct formula for $+$ in the other ring. It is conjectured that no (other) equational and logical definitions of $+$ for $p = 2$ and $p = 3$ exist which cover each other, and similar conjectures seem reasonable for p-rings and p'-rings, for $p \neq p'$.

Exactly as the logic of propositions ($=$ 2-valued logic) is mathematically equivalent to the simplest 2-ring ($=$ field-logic F_2), so the 3-valued logic is mathematically equivalent to the simplest 3-ring, the field-logic F_3. In 3-valued logic the "logical language" is $(F_3, \times, \times', \times'', {}^\wedge, {}^\vee)$, while the "ring language" is $(F_3, +, \times)$. The tri-ality theorem reigns over both of these, as it does over any 3-ring-logic, and throws new light on the 3-valued logic. We cannot enter into the details of this here.

One of a number of basic questions opened up by the foregoing theory and not yet completely settled, is the following: Given a ring R, is there at least one group K which is *fully adapted* to R, that is, which will convert R into a ring-logic (K)?

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ON A THEOREM OF DIEUDONNÉ

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1. The fact that a measure preserving transformation may be decomposed into ergodic parts was proved first by von Neumann, under the assumption that the domain of the transformation is an m-space (= a complete, separable, metric space with a regular measure).[1] Several years later I defined the concept of direct sum for measure spaces and, in terms of this concept, proved a more general decomposition theorem for measure spaces subject to certain separability restrictions.[2] My proof rested on a lemma, stated by Doob, concerning the existence of certain measures on such measure spaces. It was recently pointed out by Dieudonné that Doob's lemma and my decomposition theorem are both false.[3] The situation, in greater detail, is that my proof of the decomposition theorem is valid for a measure space if and only if the conclusion of Doob's lemma is correct for that space. An examination of Doob's proof shows that his conclusion is certainly correct if the measure space in question is the unit interval. Since, however, there is a wide class of measure spaces (called normal) for which it is known that they may be put into one to one measure preserving correspondence with the unit interval,[4] a large part of my original decomposition theorem is saved—all that has to be done is to replace the original separability assumption by the assumption of normality.